A proof of the asymptotic validity of a test for perfect aggregation*

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Abstract

An asymptotic proof is presented for a test of perfect aggregation in linear models developed in Pesaran et al. (1989). The limiting distribution is derived by letting the degree of disaggregation increase without bound for a fixed sample size.

1 Introduction

In a recent paper, Pesaran et al. (1989), henceforth PPK, propose a new test of the hypothesis of perfect aggregation in the context of a linear disaggregate model. It is shown there that, when the covariance matrix of the disaggregate model is known, then this test statistic is distributed as χ^2_n, where n is the number of observations. However, when the covariance matrix is unknown, the exact distribution of the statistic is not easily computable. The purpose of this paper is to show that in this case the test is still valid asymptotically. Since the test statistic has dimension n, the usual asymptotic theory which lets n, the sample size, tend to infinity is clearly not applicable. Instead we derive a limiting distribution by allowing the degree of disaggregation denoted by m to increase without bound. Similar large m-asymptotics have been used previously by Powell and Stoker (1985) and Granger (1987). The perfect aggregation test is derived in section2. Section 3 presents a proof of the asymptotic validity of the test for the special case where the covariance matrix is diagonal.

2 The perfect aggregation test

Let the disaggregate model be written as

$$H_d: \quad \mathbf{y}_i = \mathbf{X}_i \boldsymbol{\beta}_i + \mathbf{u}_i, \quad i = 1, 2, \ldots, m. \tag{1}$$

where $\mathbf{y}_i$ is an $n \times 1$ vector of n observations on the ith micro-unit, $\mathbf{X}_i$ is an $n \times k$ matrix of n observations on k regressors for the ith micro-unit, $\mathbf{u}_i$ is an $n \times 1$ vector of associated disturbances. The disturbances $\mathbf{u}_i$ are assumed to be distributed independently of $\mathbf{X}_i$ with $\mathrm{E}(\mathbf{u}_i) = \mathbf{0}$ and $\mathrm{E}(\mathbf{u}_i \mathbf{u}_j') = \sigma_{ij} \mathbf{I}_n$.

The aggregate model is given by

$$H_a: \quad \mathbf{y}_a = \mathbf{X}_a \mathbf{b} + \boldsymbol{\nu}_a, \tag{2}$$

where

$$\mathbf{y}_a = \sum_{i=1}^{m} \mathbf{y}_i \quad \text{and} \quad \mathbf{X}_a = \sum_{i=1}^{m} \mathbf{X}_i.$$

Then the hypothesis of perfect aggregation is defined by

$$H_\xi: \quad \xi = \sum_{i=1}^{m} \mathbf{X}_i \boldsymbol{\beta}_i - \mathbf{X}_a \mathbf{b} = \mathbf{0}, \tag{3}$$

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in which case the regression functions of (1) and (2) coincide. It can be seen that this hypothesis encompasses the two special cases of 'micro-homogeneity' ($\boldsymbol{\beta}_1 = \boldsymbol{\beta}_2 = \cdots = \boldsymbol{\beta}_m$) and 'compositional stability' ($\mathbf{X}_i = \mathbf{X}_a \mathbf{C}_i, i = 1, 2, \ldots, m$) but it can also be satisfied in more general cases (see PPK for details).

A test of the hypothesis (3) can be constructed based on OLS estimates of $\boldsymbol{\beta}_i$. Let

$$\widehat{\xi} = \sum_{i=1}^{m} \mathbf{X}_i \widehat{\boldsymbol{\beta}}_i - \mathbf{X}_a \widehat{\mathbf{b}} = \mathbf{e}_a - \mathbf{e}_d, \tag{4}$$

where the hat denotes OLS estimates,

$$\mathbf{e}_a = (\mathbf{I}_n - \mathbf{A}_a)\mathbf{y}_a,$$
$$\mathbf{e}_d = \sum_{i=1}^{m} (\mathbf{I}_n - \mathbf{A}_i)\mathbf{y}_i = \sum_{i=1}^{m} \mathbf{e}_i,$$
$$\mathbf{A}_a = \mathbf{X}_a (\mathbf{X}_a' \mathbf{X}_a)^{-1} \mathbf{X}_a',$$
$$\mathbf{A}_i = \mathbf{X}_i (\mathbf{X}_i' \mathbf{X}_i)^{-1} \mathbf{X}_i' = \mathbf{I}_n - \mathbf{M}_i.$$

The test statistic is then given by

$$\underline{a}_m = m^{-1} (\mathbf{e}_a - \mathbf{e}_d)' \widehat{\boldsymbol{\Psi}}_m^{-1} (\mathbf{e}_a - \mathbf{e}_d),$$

$$\widehat{\boldsymbol{\Psi}}_m = m^{-1} \sum_{i,j=1}^{m} \widehat{\sigma}_{ij} \mathbf{H}_i \mathbf{H}_j, \tag{5}$$
$$\mathbf{H}_i = \mathbf{A}_i - \mathbf{A}_a,$$
$$\widehat{\sigma}_{ij} = \{n - 2k + \mathrm{tr}(\mathbf{A}_i \mathbf{A}_j)\}^{-1} \mathbf{e}_i' \mathbf{e}_j.$$

It is shown in PPK that $\widehat{\sigma}_{ij}$ is an unbiased estimator of the covariance element σ_{ij}.

3 A proof of the asymptotic validity of the test

In this section a proof is presented of the asymptotic validity of the test of perfect aggregation for the special case where the disturbances u_{it} are distributed independently across equations so that $\sigma_{ij} = 0, i \neq j$. The framework adopted is to let the degree of disaggregation, m, increase without bound while keeping the sample size, n, fixed.

The following assumptions are made:

Assumption 1. The standardised micro-disturbances $\nu_{it} = u_{it}/\sqrt{\sigma_{ii}}$ are identically distributed, independently both across time periods and across equations, with zero means, unit variances and finite third-order moments.[1]

Assumption 2. The average matrix $\overline{\mathbf{X}}_m = m^{-1}\mathbf{X}_a$, and the aggregate projection matrix $\overline{\mathbf{X}}_m(\overline{\mathbf{X}}_m'\overline{\mathbf{X}}_m)^{-1}\overline{\mathbf{X}}_m'$, converge (in probability) to finite limits.

Assumption 3. The elements of the disaggregate projection matrices, $\mathbf{A}_i = \mathbf{X}_i(\mathbf{X}_i'\mathbf{X}_i)^{-1}\mathbf{X}_i'$, remain bounded in absolute value as $m \to \infty$. Notationally, we write $|\mathbf{A}_i| < \mathbf{P} < \infty$.

Assumption 4. The elements of the variance matrix $\boldsymbol{\Sigma} = (\sigma_{ij})$ remain bounded $m \to \infty$. Namely, $|\sigma_{ij}| < \tau^2 < \infty, \forall i, j$.

Assumption 5. The variance matrix $\boldsymbol{\Psi}_m$ defined by

$$\boldsymbol{\Psi}_m = m^{-1} \sum_{i,j=1}^{m} \sigma_{ij} \mathbf{H}_i \mathbf{H}_j,$$

[1] The assumption that ν_{it} have finite third-order moments can be replaced by the slightly weaker assumption that, for some positive δ, $\mathrm{E}\,|\nu_{it}|^{2+\delta}$ is uniformly bounded. See, for example, (White, 1984, Theorem 5.10).

tends to a non-singular matrix $\mathbf{\Psi}$, as $m \to \infty$.

Theorem 1. Under *Assumptions 1—5* and conditional on $\mathbf{X}$, the statistic

$$\underline{a}_m = (\mathbf{e}_a - \mathbf{e}_d)' \left(\sum_{i=1}^{m} \widehat{\sigma}_{ii} \mathbf{H}_i^2 \right)^{-1} (\mathbf{e}_a - \mathbf{e}_d)$$

will be asymptotically distributed as a χ_n^2 variate on the null hypothesis of perfect aggregation (3), as $m \to \infty$.

Proof. Let

$$\mathbf{G}_m = \left(\sum_{i=1}^{m} \widehat{\sigma}_{ii} \mathbf{H}_i^2 \right)^{-1/2} (\mathbf{e}_a - \mathbf{e}_d). \tag{6}$$

Then the test statistic in the theorem can be written as

$$\underline{a}_m = \mathbf{G}_m' \mathbf{G}_m. \tag{7}$$

Consider now the probability limit of $\widehat{\mathbf{\Psi}}_m = m^{-1} \sum_{i=1}^{m} \widehat{\sigma}_{ii} \mathbf{H}_i^2$, as $m \to \infty$. Under (1) we obtain

$$\widehat{\mathbf{\Psi}}_m = [m(n-k)]^{-1} \sum_{i=1}^{m} (\mathbf{u}_i' \mathbf{M}_i \mathbf{u}_i) \mathbf{H}_i^2. \tag{8}$$

But, since $\mathbf{M}_i$ is an idempotent matrix of rank $n-k$, we can also write

$$\sigma_{ii}^{-1} \mathbf{u}_i' \mathbf{M} \mathbf{u}_i = \sum_{t=1}^{n-k} \epsilon_{it}^2, \quad i = 1, 2, \ldots, m, \tag{9}$$

where ϵ_{it} represents scalar random variables distributed independently across i and t with zero means and unit variances. Substituting (9) in (8) yields

$$\widehat{\mathbf{\Psi}}_m = (n-k)^{-1} \sum_{t=1}^{n-k} m^{-1} \left(\sum_{i=1}^{m} \sigma_{ii} \epsilon_{it}^2 \mathbf{H}_i^2 \right). \tag{10}$$

But, noting that $\mathbf{H}_i = \mathbf{A}_i - \mathbf{A}_a$, we have

$$m^{-1} \sum_{i=1}^{m} \sigma_{ii} \epsilon_{it}^2 \mathbf{H}_i^2 = f_m \mathbf{A}_a + \mathbf{F}_m - \mathbf{F}_m \mathbf{A}_a - \mathbf{A}_a \mathbf{F}_m, \tag{11}$$

where

$$f_m = m^{-1} \sum_{i=1}^{m} \sigma_{ii} \epsilon_{it}^2, \quad \mathbf{F}_m = m^{-1} \sum_{i=1}^{m} \sigma_{ii} \epsilon_{it}^2 \mathbf{A}_i.$$

Now, under Assumption 4, it readily follows that

$$\operatorname*{plim}_{m \to \infty} (f_m) \leq \tau^2 \operatorname*{plim}_{m \to \infty} \left(m^{-1} \sum_{i=1}^{m} \epsilon_{it}^2 \right),$$

and since ϵ_{it} are identically and independently distributed random variables then, by the law of large numbers, $m^{-1} \sum_{i=1}^{m} \epsilon_{it}^2 \xrightarrow{p} 1$, and

$$\operatorname*{plim}_{m \to \infty} (f_m) \leq \tau^2 < \infty. \tag{12}$$

Similarly, under Assumptions 3 and 4, we have

$$\operatorname*{plim}_{m \to \infty} (\mathbf{F}_m) \leq \tau^2 \mathbf{P} < \infty, \tag{13}$$

where $\mathbf{P}$ is already defined by Assumption 3. The results (12) and (13) establish the existence of the probability limits of f_m and $\mathbf{F}_m$, as $m \to \infty$, and this in turn establishes [using (11) and noting that, by Assumption 2, matrix $\mathbf{A}_a$, has a finite limit as $m \to \infty$] that

$$\operatorname*{plim}_{m \to \infty} \left(m^{-1} \sum_{i=1}^{m} \sigma_{ii} \epsilon_{it}^2 \mathbf{H}_i^2 \right) = \lim_{m \to \infty} \left(m^{-1} \sum_{i=1}^{m} \sigma_{ii} \mathbf{H}_i^2 \right).$$

Using this result in (10), we finally obtain

$$\widehat{\boldsymbol{\Psi}}_m = m^{-1}\sum_{i=1}^{m}\widehat{\sigma}_{ii}\mathbf{H}_i^2 \xrightarrow{p} \lim_{m\to\infty}\left(m^{-1}\sum_{i=1}^{m}\sigma_{ii}\mathbf{H}_i^2\right) = \boldsymbol{\Psi}. \tag{14}$$

Therefore, asymptotically we have[2]

$$\mathbf{G}_m \overset{a}{\sim} \boldsymbol{\Psi}^{-1/2} m^{-1/2}(\mathbf{e}_a - \mathbf{e}_d).$$

But, under (1) and on the assumption that $H_\xi : \sum_{i=1}^{m}\mathbf{X}_i\widehat{\boldsymbol{\beta}}_i = \mathbf{X}_a\widehat{\mathbf{b}}$ holds,

$$m^{-1/2}(\mathbf{e}_a - \mathbf{e}_d) = m^{-1/2}\sum_{i=1}^{m}\mathbf{H}_i\mathbf{u}_i.$$

Hence,

$$\mathbf{G}_m \overset{a}{\sim} m^{-1/2}\sum_{i=1}^{m}\mathbf{z}_i, \tag{15}$$

in which

$$\mathbf{z}_i = \sqrt{\sigma_{ii}}\boldsymbol{\Psi}^{-1/2}\mathbf{H}_i\boldsymbol{\nu}_i,$$

and $\boldsymbol{\nu}_i = \mathbf{u}_i/\sqrt{\sigma_{ii}}$. We now show that under the assumptions of the theorem, as $m \to \infty$, the sum $\mathbf{s}_m = m^{-1/2}\sum_{i=1}^{m}\mathbf{z}_i$ tends to a multivariate normal distribution with mean zero and covariance matrix $\mathbf{I}_n$, an identity matrix of order n. For this purpose, it is sufficient to demonstrate that for any fixed vector $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_n)'$, the limiting distribution of $\boldsymbol{\lambda}'\mathbf{s}_m$ is $N(0, \boldsymbol{\lambda}'\boldsymbol{\lambda})$.

Let

$$d_m = \boldsymbol{\lambda}'\mathbf{s}_m = m^{-1/2}\sum_{i=1}^{m} w_i, \tag{16}$$

in which

$$w_i = \sqrt{\sigma_{ii}}\boldsymbol{\lambda}'\boldsymbol{\Psi}^{-1/2}\mathbf{H}_i\boldsymbol{\nu}_i, \quad i = 1, 2, \ldots, m \tag{17}$$

is now a scalar random variable. We have, for all i,

$$\mathrm{E}(w_i) = 0,$$

$$\mathrm{V}(w_i) = \sigma_{ii}\boldsymbol{\lambda}'\boldsymbol{\Psi}^{-1/2}\mathbf{H}_i^2\boldsymbol{\Psi}^{-1/2}\boldsymbol{\lambda} > \mathbf{0}.$$

Setting $\boldsymbol{\mu} = \boldsymbol{\Psi}^{-1/2}\boldsymbol{\lambda}$, then

$$C_m^2 = \sum_{i=1}^{m}\mathrm{V}(w_i) = \boldsymbol{\mu}'\left(\sum_{i=1}^{m}\sigma_{ii}\mathbf{H}_i^2\right)\boldsymbol{\mu}. \tag{18}$$

Denoting the (s,t) element of matrix $\mathbf{H}_i$ by $h_{i,st}$, we also have [using (17)]

$$w_i = \sqrt{\sigma_{ii}}\sum_{t=1}^{n}\left(\sum_{s=1}^{n}\mu_s h_{i,st}\right)\nu_{it}.$$

Therefore, since by assumption $\boldsymbol{\Psi}$ is non-singular and $h_{i,st}$ are bounded in absolute value for all i, then

$$|w_i| \le n\kappa\sqrt{\sigma_{ii}}\left|\sum_{t=1}^{n}\nu_{it}\right|,$$

where $|\mu_s h_{i,st}| < \kappa < \infty$. Consequently,

$$\mathrm{E}\,|w_i|^3 \le n^3\kappa^3\sigma_{ii}^{3/2}\,\mathrm{E}\left|\sum_{t=1}^{n}\nu_{it}\right|^3.$$

However, since the random variables ν_{it} are i.i.d. with finite third-order moments, $\mathrm{E}\,|\sum_{t=1}^{n}\nu_{it}|^3 \le n\theta^3$, where $\theta^3 = \mathrm{E}\,|\nu_{it}|^3$, and

$$\mathrm{E}\,|w_i|^3 \le n^4\kappa^3\theta^3\sigma_{ii}^{3/2}. \tag{19}$$

[2] Note that, by Assumption 5, matrix $\boldsymbol{\Psi}$ is non-singular.

We are now in a position to apply the Liapunov Central Limit Theorem to the sum d_m defined by (16).[3] Setting

$$B_m^3 = \sum_{i=1}^{m} \mathrm{E}\,|w_i|^3,$$

then using (19) it follows that

$$B_m^3 \leq (n^4 \kappa^3 \theta^3) \sum_{i=1}^{m} \sigma_{ii}^{3/2},$$

which together with (18) yields[4]

$$\lim_{m\to\infty} \left[\frac{B_m}{C_m}\right] \leq \left[\frac{n^{4/3}\kappa\theta}{(\boldsymbol{\lambda}'\boldsymbol{\lambda})^{1/2}}\right] \lim_{m\to\infty} m^{-1/2} \left[\sum_{i=1}^{m} \sigma_{ii}^{3/2}\right]^{1/3}.$$

But, under Assumption 4,

$$\lim_{m\to\infty} m^{-1/2} \left[\sum_{i=1}^{m} \sigma_{ii}^{3/2}\right]^{1/3} \leq \lim_{m\to\infty} (m^{-1/6}\tau) = 0,$$

and for a fixed n, we have $\lim(B_m/C_m) = 0$, as $m \to \infty$, and the condition of the Liapunov theorem will be met. Hence,

$$\mathbf{G}_m \overset{a}{\sim} \mathbf{s}_m \overset{a}{\sim} N(\mathbf{0}, \mathbf{I}_n).$$

Now, using (7), we have

$$\underline{a}_m = \mathbf{G}_m' \mathbf{G}_m \overset{a}{\sim} \chi_n^2.$$

Q.E.D.

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[3] See, for example, (Rao, 1973, p. 127).

[4] Notice that $\lim_{m\to\infty}\{\boldsymbol{\mu}'(m^{-1}\sum_{i=1}^{m}\sigma_{ii}\mathbf{H}_i^2)\boldsymbol{\mu}\} = \boldsymbol{\mu}'\boldsymbol{\Psi}\boldsymbol{\mu} = \boldsymbol{\lambda}'\boldsymbol{\lambda}$.