LINEAR PROGRAMMING 1

In many business and policy making situations the following type of problem is encountered:

Maximise an objective subject to (in)equality constraints.

Mathematical programming provides techniques by which limited resources may be allocated optimally among various activities. The simplest and most widely applied of these is Linear Programming. The basic ideas were developed by L.V. Kantarovich, a Russian mathematician, in the late 1930s but the computational procedure is due to George Dantzig, a US mathematician working on air force scheduling problems in World War II. Before studying the method we should note that LP problems must have:

1. **a well-specified objective function** e.g. maximising profits or minimising costs

2. **alternative courses of action** must be possible.

3. all resource constraints and objectives must be in **linear** form.
   A function $f(x) = f(x1, x2, ..., xn)$ is linear if
   $f(x1,x2,...,xn) = a1\ x1 + a2\ x2 + ... + an\ xn$
   for any constant parameters a1, ..., an. Note that intercepts are ruled out and this assumption implies constant returns to scale

4. resources must be limited in supply - the traditional assumption of scarcity.

AN EXAMPLE:

$$\max\ 8X_1 + 10X_2$$

subject to

$$4.5X_1 + 2X_2 \le 36 \qquad (1)$$

$$3X_1 + 5X_2 \le 45 \qquad (2)$$

$$4X_1 + 4X_2 \le 40 \qquad (3)$$

$$X_1, X_2 \ge 0$$

Introduce *slack variables* S1, S2, S3 to transform inequalities to equalities. Problem becomes:

$$\max\ 8X_1 + 10X_2 + 0S_1 + 0S_2 + 0S_3$$

subject to

$$4.5X_1 + 2X_2 + S_1 = 36 \qquad (1')$$

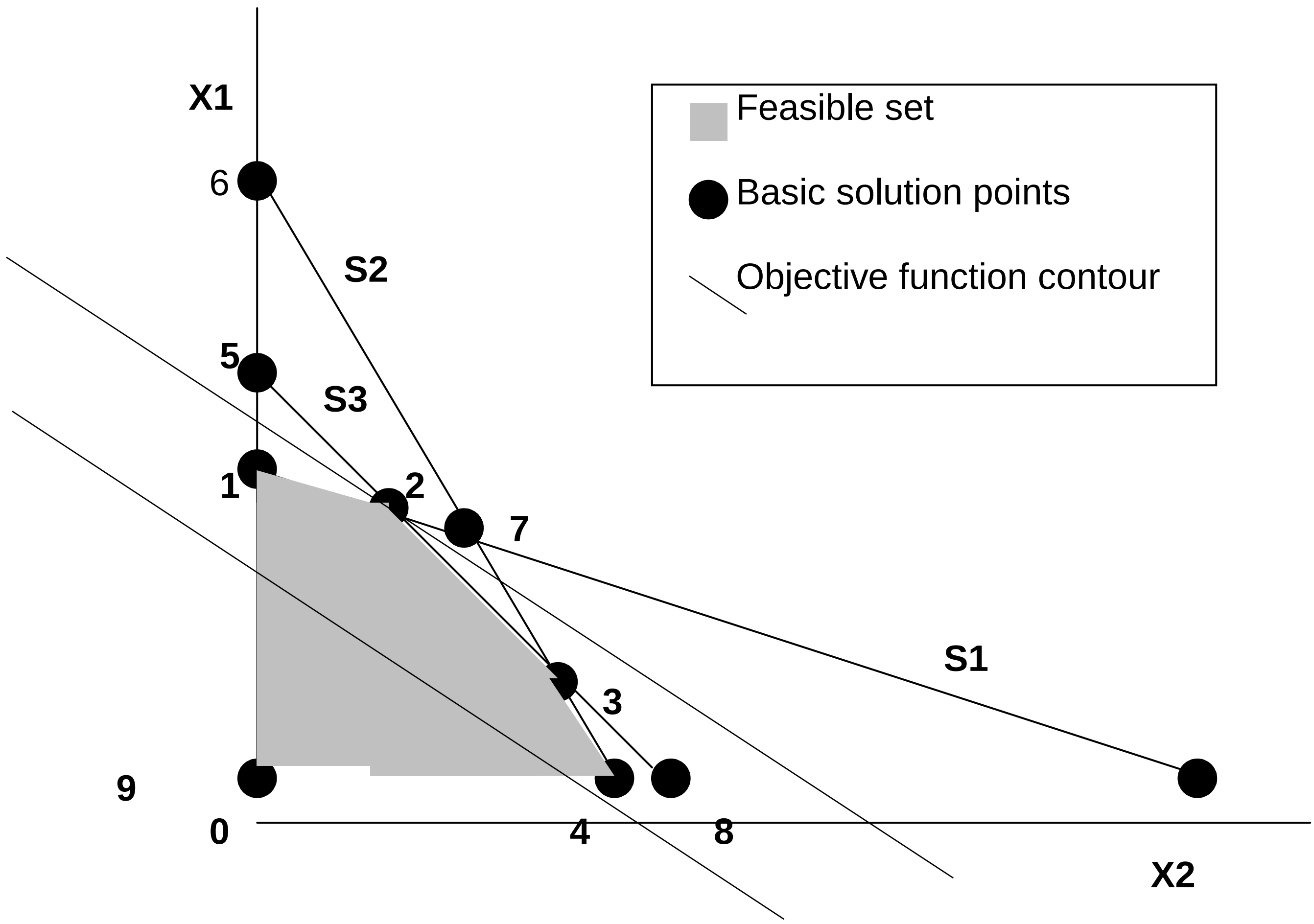
$$3X_1 + 5X_2 + S_2 = 45 \qquad (2')$$

$$4X_1 + 4X_2 + S_3 = 40 \qquad (3')$$

There are 5 variables: $X_1, X_2, S_1, S_2, S_3 \ge 0$

There are 3 constraints: 1', 2', 3'

Graphical Representation (Maximisation Problem)

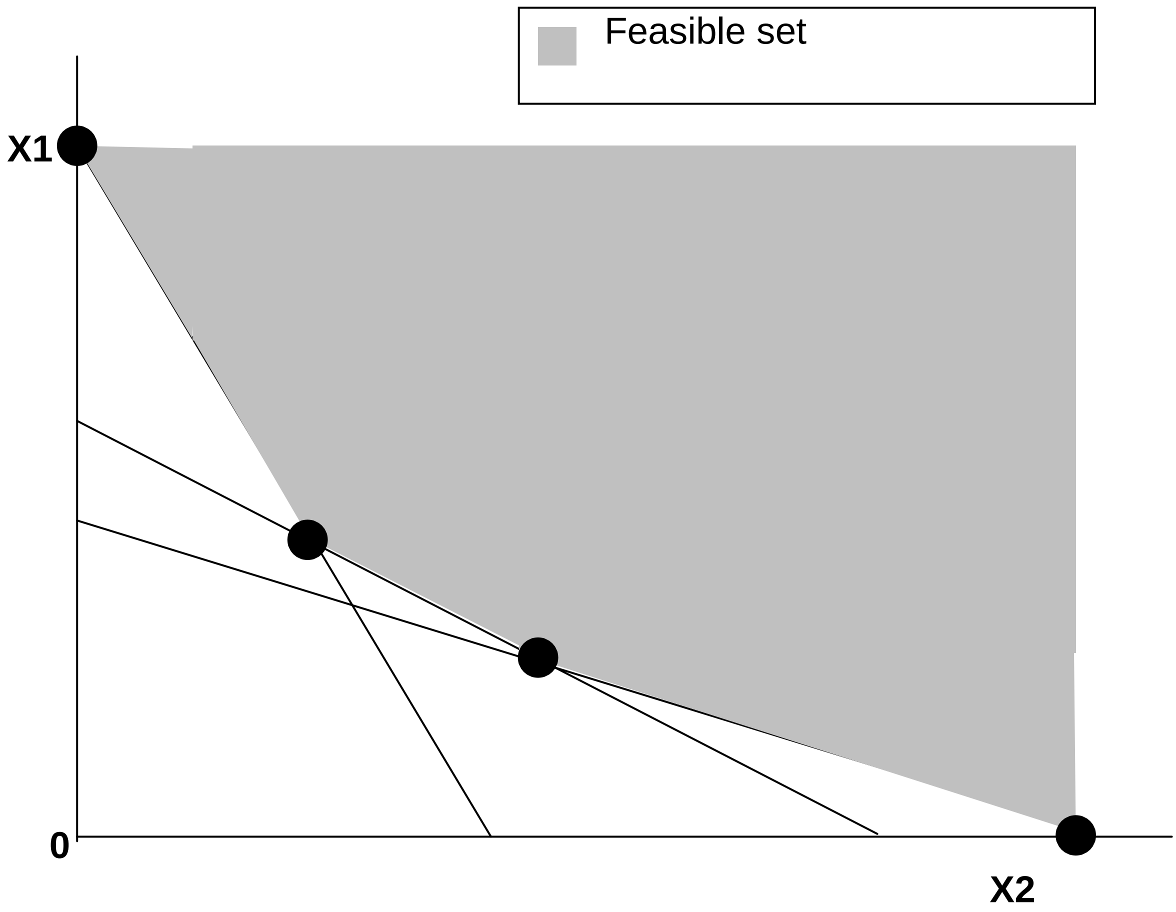


At any Basic Solution point:
exactly 3 variables will be non-zero and 2 will be zero

0	**S1, S2, S3 > 0**	**5**	**X1, S2 > 0, S1 < 0**
1	**X1, S2, S3 > 0**	**6**	**X1 > 0, S1, S2 < 0**
2	**X1, X2, S2 > 0**	**7**	**X1, X2 > 0, S3 < 0**
3	**X1, X2, S1 > 0**	**8**	**X2, S1 > 0, S2 < 0**
4	**X2, S1, S3 > 0**	**9**	**X2 > 0, S2, S3 < 0**

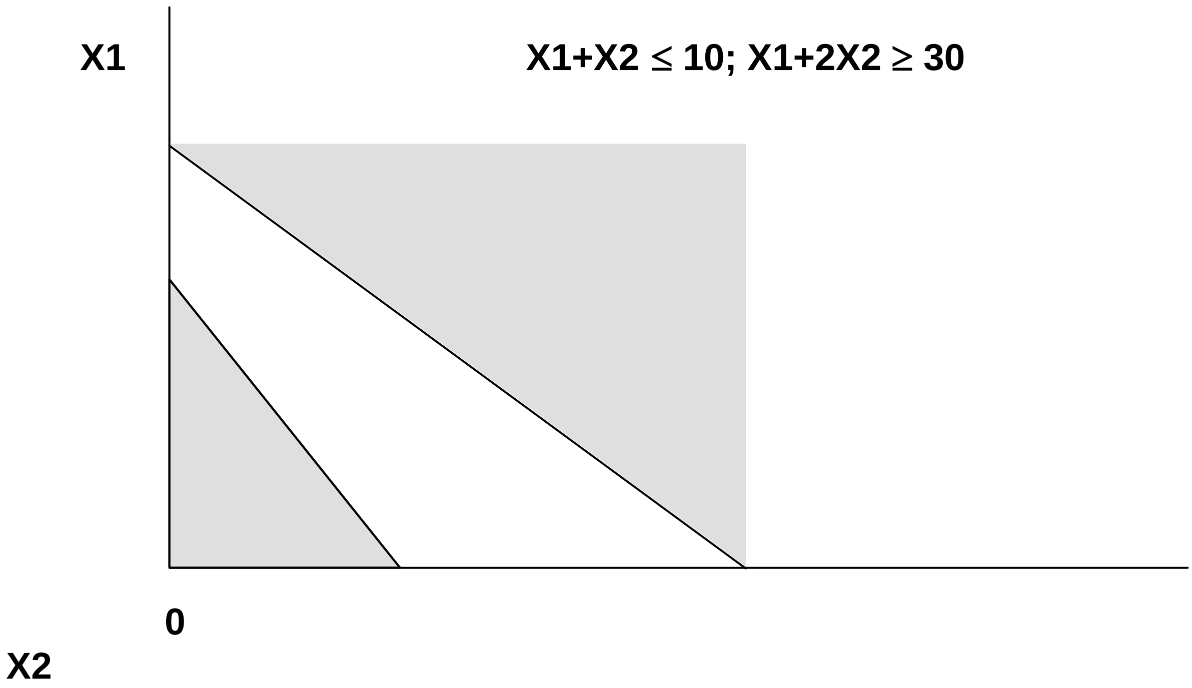
Points 0-4 are in feasible set. Points 5-9 are not.

Graphical Representation (Minimisation Problem)

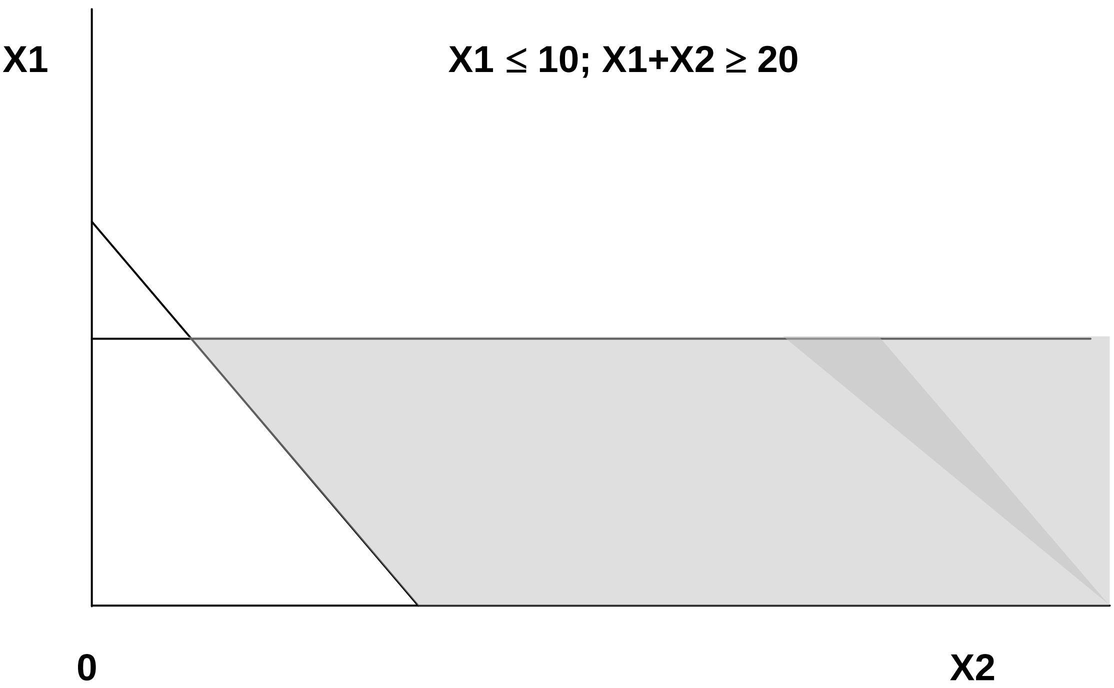


Problems that Can't be Solved

Infeasible Problem



Unbounded Problem



DEFINITIONS

The Feasible Set

The feasible set is the set of all points that satisfy all the constraints of the problem.

Basic Solution

A basic solution is a point where the number of non-zero variables is equal to the number of constraints

Basic Variable

A basic variable is one that is non-zero in a basic solution. Zero variables are non basic.

The Fundamental Theorem of Linear Programming

An optimal solution can be found by considering only basic solutions.

=> optimal solution will be at corner (extreme) point

Shadow Price of Resource

How much does the objective function change if 1 more (or 1 less) unit of a resource is available?

The shadow price of a resource constraint is only non-zero when the constraint is binding.

DUALITY : The Primal Problem

max b1 X1 + b2 X2

subject to

a11 X1 + a12 X2 $\leq$ c1 (1)

a21 X1 + a22 X2 $\leq$ c2 (2)

a31 X1 + a32 X2 $\leq$ c3 (3)

X1, X2 $\geq$ 0

2 Variables: X1, X2
3 $\leq$ Constraints: (1), (2), (3)

DUALITY: The Dual Problem

min c1 Y1 + c2 Y2 + c3 Y3

subject to

a11 Y1 + a21 Y2 + a31 Y2 $\geq$ b1 (4)

a12 Y1 + a22 Y2 + a32 Y2 $\geq$ b2 (5)

Y1, Y2, Y3 $\geq$ 0

3 Variables: Y1, Y2, Y3
2 $\geq$ Constraints: (4), (5)

What is the interpretation of variables:Y1,Y2,Y3 ?

Y1,Y2,Y3 correspond to shadow prices of the resource constraints in the primal problem.

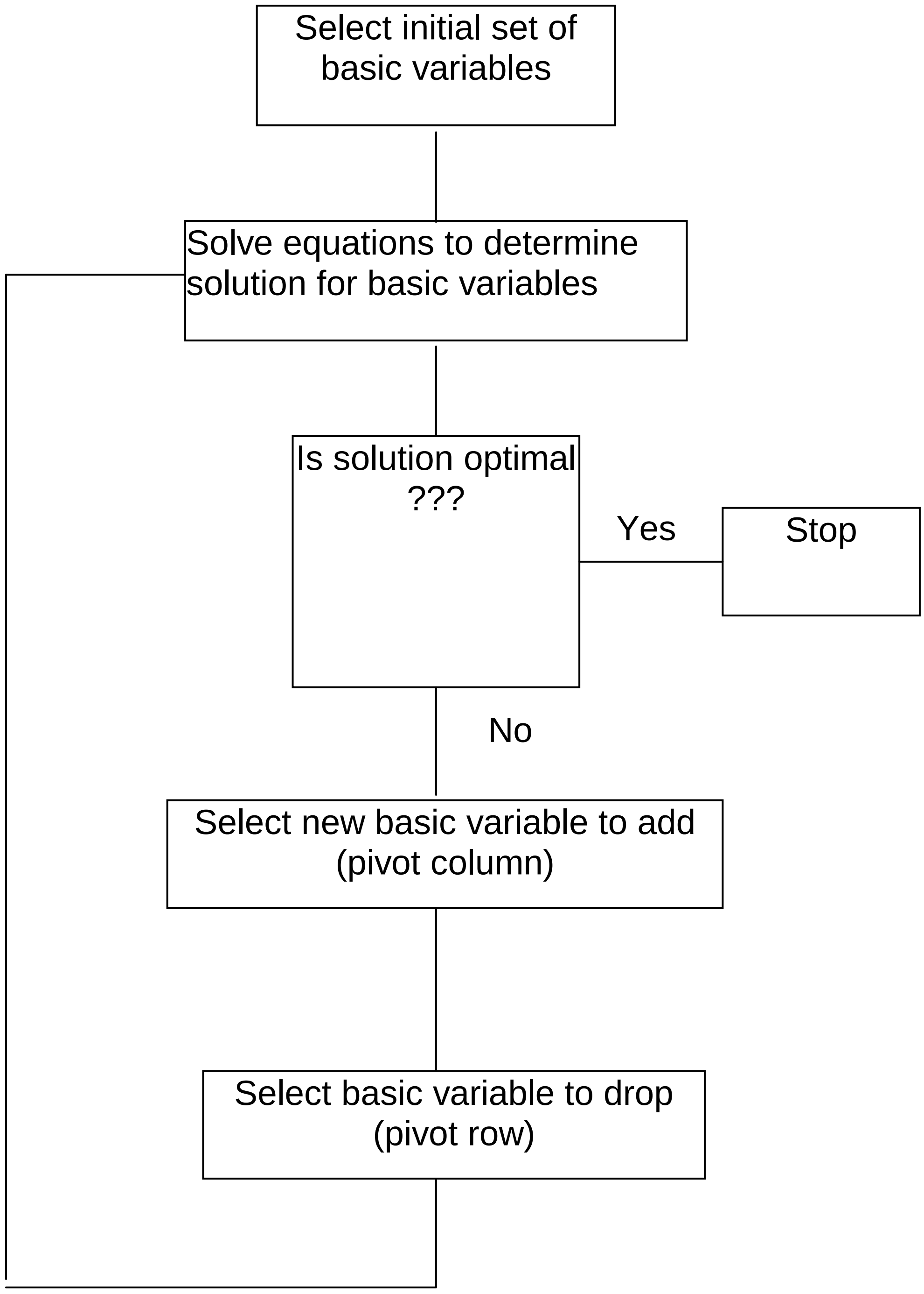
This implies that for every profit maximisation problem there is a unique set of prices which exactly distribute profits to the fixed constraints. These prices can be used for investment decision making since they show the value of marginal changes in resource availability.

Using the Software

There are two LP programs. the first is a graphical display restricted to 2 variables only. Nevertheless it can handle a large number of constraints and provides useful information on the nature of the problem. The Data Entry Method provides a choice of free form or tabular input and it is simplest to choose Free Form initially. A request for Objective Function data comes next: select MAX or MIN depending on the problem. Then enter the objective function directly e.g. for the example problem 8 X1+10 X2. Note that no * is required for multiplication. Each constraint is identified by a number, so that constraint 1 is written: 1 4.5 X1 + 2 X2 <= 36, constraint 2 is written as 3X1+5X2<=45 and 3 as 3 4X1+4X2<=40. Note that $\leq$ constraints are specified using the character pair <= and $\geq$ as the character pair >=.

There are only 3 constraints so that when constraint 4 is requested simply type GO. The program asks whether you want to modify entries or to proceed to solution. When you are ready to proceed, a graphical presentation appears which enables the user to find the optimal solution by moving the objective function line within the feasible region. the Basic Solution Points can be highlighted (F5) as can the optimal point (F6) and the program offers limited re-scaling and re-drawing features. Solution values can be listed together with slack values for each constraint. Extreme points can be listed (10 in the example). An ending menu enables modification of data, printing of results or saving data to a file. A printed file (ASCII) can be input into a word processing package to create a report.

The Simplex Algorithm



A Simplex Tableau

Cj		8	10	0	0	0	
	Variable	X1	X2	S1	S2	S3	RHS
0	S1	4.5	2	1	0	0	36
0	S2	3	5	0	1	0	45
0	S3	4	4	0	0	1	40
	Zj	0	0	0	0	0	0
	Cj-Zj	8	10	0	0	0	

1 The columns of the simplex tableau represent the variables in the problem plus the right hand side of the constraints.

2 The first row Cj gives the coefficients in the objective function.

3 The subsequent (shaded) rows give for each basic (non-zero) variable the coefficients in the corresponding equation.

4 The columns of the tableau represent how much of the basic variables are required to produce 1 unit of the column variable.

5 The Zj row gives for each column variable, the opportunity cost of producing one extra unit of that variable defined by

$$Zj = \sum_i Ci\ aij$$

6 The final Cj-Zj row gives the net improvement to the objective function of producing one unit of variable j. If any of the elements of this row are positive, then the solution can be improved by substituting variable j for one of the current basic variables. The variable to add is the one with the *largest value of Cj-Zj*. The corresponding column is known as the **pivot column**.

7 As more of the new variable is added eventually a constraint will become binding and one of the basic variables will go to zero. The row for this variable then disappears from the tableau. The variable that disappears is the one for which the *ratio of RHS / pivot column is smallest*. The corresponding row is called the **pivot row**.

8 The intersection of pivot column and pivot row is called the **pivot element**.

Simplex Tableau 1

Cj		8	10	0	0	0	
	Variable	X1	X2	S1	S2	S3	RHS
0	S1	4.5	2	1	0	0	36
0	S2	3	5	0	1	0	45
0	S3	4	4	0	0	1	40
	Zj	0	0	0	0	0	0
	Cj-Zj	8	10	0	0	0	

There are positive elements in Cj-Zj column so solution not optimal.
Pivot column is X2 which has highest Cj-Zj (10)
Pivot row is S2 which has smallest RHS / Pivot column (45/5)

Solving the equations of Tableau 1
Step 1: Divide pivot row by pivot element (5)

	Variable	X1	X2	S1	S2	S3	RHS
* 0.2	X2	0.6	1	0	0.2	0	9

Step 2: Subtract pivot row * pivot column element from each other row

	Variable	X1	X2	S1	S2	S3	RHS
S1	S1	4.5	2	1	0	0	36
-2*X2	2*X2	1.2	2	0	0.4	0	18
=	S1	3.3	0	1	-0.4	0	18

	Variable	X1	X2	S1	S2	S3	RHS
S3	S3	4	4	0	0	1	40
-4*X2	4*X2	2.4	4	0	0.8	0	36
=	S3	1.6	0	0	-0.8	1	4

Simplex Tableau 2

Cj		8	10	0	0	0	
	Variable	X1	X2	S1	S2	S3	RHS
0	S1	3.3	0	1	-0.4	0	18
10	X2	0.6	1	0	0.2	0	9
0	S3	1.6	0	0	-0.8	1	4
	Zj	6	10	0	2	0	90
	Cj-Zj	2	0	0	-2	0	

There are still positive elements in Cj-Zj column so not optimal.
Pivot column is X1 which has highest Cj-Zj (2)
Pivot row is S3 which has smallest RHS / Pivot column (4/1.6)

Solving the equations of Tableau 2
Step 1: Divide pivot row by pivot element (1.6)

	Variable	X1	X2	S1	S2	S3	RHS
* 0.625	X1	1	0	0	-0.5	0.625	2.5

Step 2: Subtract pivot row * pivot column from each other row

	Variable	X1	X2	S1	S2	S3	RHS
S1	S1	3.3	0	1	-0.4	0	18
-3.3*X2	3.3*X1	3.3	0	0	-1	1.25	8.25
=	S1	0	0	1	1.25	0	9.75

	Variable	X1	X2	S1	S2	S3	RHS
X2	X2	0.6	1	0	0.2	0	9
-0.6*X2	0.6*X1	0.6	0	0	-0.3	0.375	1.5
=	X2	0	1	0	0.5	-0.375	7.5

Simplex Tableau 3

Cj		8	10	0	0	0	
	Variable	X1	X2	S1	S2	S3	RHS
0	S1	0	0	1	1.25	0	9.75
10	X2	0	1	0	0.5	-0.375	7.5
8	X1	1	0	0	-0.5	0.625	2.5
	Zj	8	10	0	1	1.25	95
	Cj-Zj	0	0	0	-1	-1.25	

This is optimal since all Cj-Zj are ≤ 0. Solution: X1=2.5, X2=7.5;