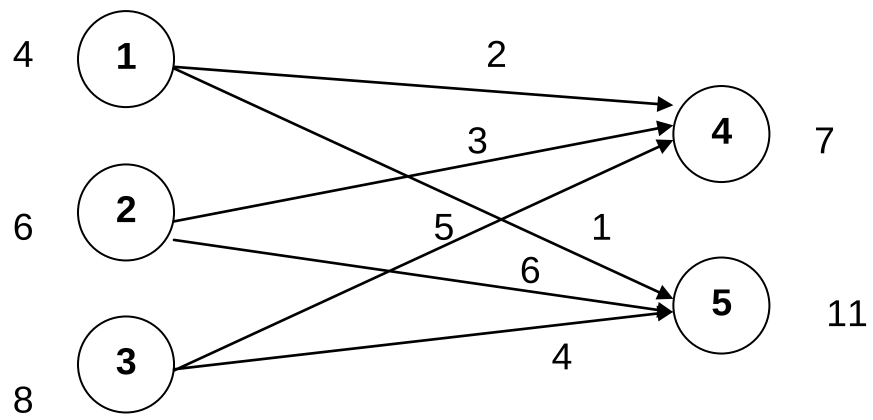
TRANSPORTATION & NETWORK PROBLEMS

Transportation Problems

Problem: moving output from multiple sources to multiple destinations. The objective is to minimise costs (maximise profits).

Network Representation

Transportation problems have a natural network representation.



In the diagram there are 3 sources represented by nodes **1** - **3** and 2 destinations represented by nodes **4** and **5**.

The arrows show flows of output from source to destination. Each destination is linked to each source by an arrow. The number above each arrow represents the cost of transporting on that route.

The supply at each source is represented by the number next to each source node. Similarly the demand at each destination is represented by the number next to the destination node.

Note that total supply 4 + 6 + 8 equals total demand 7 + 11. A problem in which demand equals supply is called a *balanced problem*.

The transportation problem to be solved is to determine how much should travel from each source node to each destination node in order to minimise costs.

Linear Programming Formulation

Let **Xij** represent output moving from source **i** to destination **j**.
Let **Cij** be the cost of transporting output from **i** to **j**.

Then the transportation problem can be written as

$$\min \sum_{i} \sum_{j} C_{ij} X_{ij}$$

subject to

$$\sum_{i} X_{ij} \geq D_j \quad \text{(demand restriction for node j)} \quad (1)$$

$$\sum_{j} X_{ij} \leq S_i \quad \text{(supply restriction for node i)} \quad (2)$$

$$\sum_{i} S_i = \sum_{j} D_j \quad \text{(demand = supply)} \quad (3)$$

This is a standard Linear Programming problem although when the number of sources is m and the number of destinations is n then the number of restrictions is n + m + 1 which quickly becomes large.

Example: 3 sources and 2 destinations

$$\min \quad 2 X_{14} + 1 X_{15} + 3 X_{24} + 6 X_{25} + 5 X_{34} + 4 X_{35}$$

subject to

$$X_{14} + X_{24} + X_{34} \geq 7 \quad (1) \quad \text{(demand)}$$
$$X_{15} + X_{25} + X_{35} \geq 11 \quad (2)$$

$$X_{14} + X_{15} \leq 4 \quad (3) \quad \text{(supply)}$$
$$X_{24} + X_{25} \leq 6 \quad (4)$$
$$X_{34} + X_{35} \leq 8 \quad (5)$$

$$7 + 11 = 4 + 6 + 8 \quad (6) \quad \text{(demand = supply)}$$

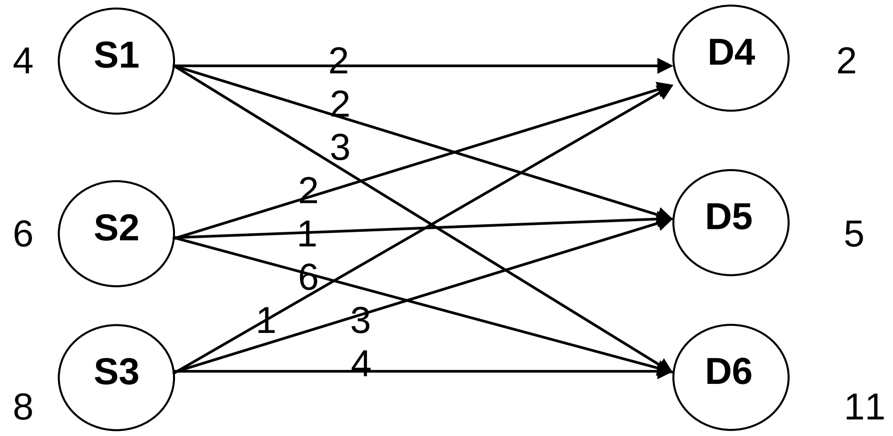
In this problem total demand = total supply so the final restriction is satisfied trivially. If not then the problem is said to be **unbalanced**. An unbalanced problem can be balanced by adding a dummy source or destination with zero associated transportation costs.

Efficient Solution Methods

Transportation problems can be solved using standard Linear Programming techniques. However, the special nature of transportation problems with their adding up restrictions means that
specialised solution algorithms can be devised that are far more efficient than the simplex method.

The Stepping Stone Algorithm

Consider the transportation problem in the network diagram:



This problem can also be represented in a tabular form as below. The rows represent the sources and the columns the destinations. The final column shows the total supply from each source and the final row the total demand to each destination.

	D4	D5	D6	Supply
S1	2)	2)	3)	4
S2	2)	1)	6)	6
S3	1)	3)	4)	8
Demand	2	5	11	18

The parenthesised numbers in each cell show the cost of transportation on the route Si to Dj.

The problem is to allocate output to each cell in the table so as to minimise total cost while satisfying the demand and supply restrictions that:

1) the columns must sum to the final total supply column
2) the rows must sum to the final total demand row

An Initial Allocation: the North West corner method

The North West corner method is a simple method of making an arbitrary initial allocation that satisfies the adding up constraints of the problem. Starting from the top left corner cell (North West), allocate as much as possible to that cell. This will be the minimum of the supply column and demand row values. Then the other cells in the column (row) must be zero to satisfy the adding up restriction for that column (row)

	D4	D5	D6	Supply
S1	2) **2**	2)	3)	4
S2	2) **0**	1)	6)	6
S3	1) **0**	3)	4)	8
Demand	2	5	11	18

Having defined the first column (row), turn to the first unallocated cell in the next column (row). Allocate as much as possible to this cell. This defines another row (column).

	D4	D5	D6	Supply
S1	2) **2**	2) **2**	3) **0**	4
S2	2) **0**	1)	6)	6
S3	1) **0**	3)	4)	8
Demand	2	5	11	18

Continue until all cells have been allocated. The complete initial allocation is shown below.

	D4	D5	D6	Supply
S1	2) **2**	2) **2**	3) **0**	4
S2	2) **0**	1) **3**	6) **3**	6
S3	1) **0**	3) **0**	4) **8**	8
Demand	2	5	11	18

The transportation cost associated with this allocation is given by

$$\sum_{i} \sum_{j} \mathbf{C_{ij} X_{ij}} = 2*2 + 2*2 + 1*3 + 6*3 + 4*8 = 61$$

Allocation 1:

	D4	D5	D6	Supply
S1	2) **2**	2) **2**	3) **0**	4
S2	2) **0**	1) **3**	6) **3**	6
S3	1) **0**	3) **0**	4) **8**	8
Demand	2	5	11	18

Having found an initial allocation we need to determine whether it can be improved or if this is the optimal allocation.

Any reallocation involves increasing output to some cells and decreasing from others. The only cells we need to consider are those currently with a zero allocation. These are the greyed cells in the diagram.

For each zero cell we need to know whether reallocating a single unit to that cell will increase or decrease cost.

Consider the zero cell S1 D6.

	D4	D5	D6	Supply
S1		2) **-**	3) **+**	4
S2		1) **+**	6) **-**	6
S3				8
Demand	2	5	11	18

- If we allocate one unit to this cell then we need to subtract one unit from one other cell in the same column (S2 D6) so that the column total (11) is unaffected.
- Similarly we must subtract one unit from one other cell in the same row (S1 D5) so that the row total (4) is unaffected.
- This in turn affects one other row and column. By adding one unit to cell S2 D5 we can reconcile both row and column totals.

The net effect on cost of the reallocation is the sum of the cost associated for each cell times ±1 as appropriate.

In this case the net effect is + 3 - 6 + 1 - 2 = - 4 so that the cost would be decreased by a reallocation.
We can make the same calculation for each zero cell in the table.

Consider the zero cell S3 D4.

	D4	D5	D6	Supply
S1	2) **-**	2) **+**		4
S2		1) **-**	6) **+**	6
S3	1) **+**		4) **-**	8
Demand	2	5	11	18

When we reallocate a unit to this cell we cannot subtract a unit from the two adjacent cells because these already have a zero allocation. Consequently the reallocation is more complicated and involves six cells (greyed).

Here the net effect is + 1 - 2 + 2 -1 + 6 - 4 = + 2 so that the cost would be increased by a reallocation.

The complete set of results for all zero cells is:

S1 D6: + 3 - 6 + 1 - 2 = - 4
S2 D4: + 2 - 2 + 2 - 1 = + 1
S3 D4: + 1 - 2 + 2 - 1 + 6 - 4 = + 2
S3 D5: + 3 -1 + 6 - 4 = + 4

Thus we see that only one cell (S1 D6) gives us a net decrease in cost. Reallocating to that cell will therefore improve the allocation. If more than one cell had implied a net decrease, the cell that offered the greatest cost reduction would have been chosen.

Note that when calculating the effect of a reallocation to an empty cell, this reallocation must involve *only* other *non-empty cells*. There are only ever m+n-1 non-empty cells in any allocation in a transportation problem and there will only ever be one unique way to do a reallocation to a zero cell.

We allocate as much as we can to cell S1 D6, which is 2 units. As a consequence cell S2 D5 is increased by 2 and cells S1 D5 and S2 D6 are decreased by 2 units.

Allocation 2:

	D4	D5	D6	Supply
S1	2) **2**	2) **0**	3) **2**	4
S2	2) **0**	1) **5**	6) **1**	6
S3	1) **0**	3) **0**	4) **8**	8
Demand	2	5	11	18

The transportation cost associated with this new allocation is given by

$$\sum\{i\} \sum\{j\} \mathbf{Cij\ Xij} = 2*2 + 3*2 + 1*5 + 6*1 + 4*8 = 53$$

Calculating the net effect on costs of reallocating to the zero cells in this new allocation we have:

S1 D5: + 2 - 3 + 6 - 1 = + 4
S2 D4: + 2 - 2 + 3 - 6 = - 3
S3 D4: + 1 - 2 + 3 - 4 = - 2
S3 D5: + 3 -1 + 6 - 4 = + 4

so that further improvement can be made by reallocating either to cell S2 D4 or cell S3 D4.

Reallocating as much as possible to route S2 D4 leads to

Allocation 3:

	D4	D5	D6	Supply
S1	2) **1**	2) **0**	3) **3**	4
S2	2) **1**	1) **5**	6) **0**	6
S3	1) **0**	3) **0**	4) **8**	8
Demand	2	5	11	18

The transportation cost associated with this new allocation is given by

$$\sum\{i\} \sum\{j\} \mathbf{Cij\ Xij} = 2*1 + 3*3 + 2*1 + 1*5 + 4*8 = 50$$

Calculating the net effect on costs of reallocating to the zero cells in allocation 3 we have:

S1 D5: + 2 - 2 + 2 - 1 = + 1
S2 D6: + 6 - 3 + 2 - 2 = + 3
S3 D4: + 1 - 2 + 3 - 4 = - 2
S3 D5: + 3 - 4 + 3 - 2 + 2 - 1 = + 1

so that further improvement can still be made by reallocating to cell S3 D4.

Reallocating as much as possible to route S3 D4 leads to

Allocation 4:

	D4	D5	D6	Supply
S1	2) **0**	2) **0**	3) **4**	4
S2	2) **1**	1) **5**	6) **0**	6
S3	1) **1**	3) **0**	4) **7**	8
Demand	2	5	11	18

The transportation cost associated with this new allocation is given by

∑{i} ∑{j} **Cij Xij** = 3*4 + 2*1 + 1*5 + 1*1 + 4*7 = 48.

In this case, no further improvement can be made so we have reached the optimal allocation.

Dealing with Prohibited Routes

We may want to prohibit transport along one or more routes. To deal with prohibited routes in the Stepping Stone algorithm, simply set transport costs on that route to an arbitrary large number M so that it will not be selected.

Dealing with Unbalanced Problems

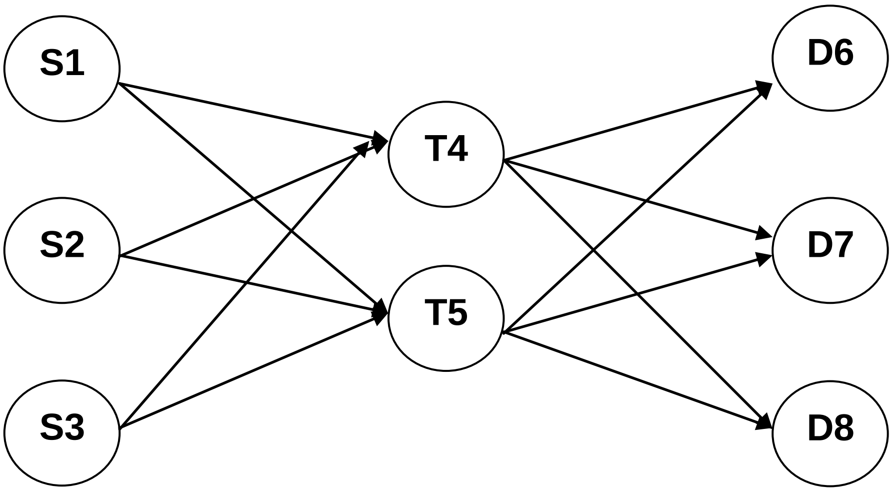
If the original transportation problem is unbalanced it needs to be balanced.

- When total demand exceeds total supply, add a dummy row to the table with supply equal to the shortfall and zero costs
- When total supply exceeds total demand, add a dummy column to the table with demand equal to the shortfall and zero costs

Transhipment Problems

Transhipment problems are more general than transportation problems. In these problems output is moved from sources to destinations through intermediary nodes called transhipment nodes.

Network representation of a transhipment problem



In the diagram nodes **T4** and **T5** are transhipment nodes. Note that all routes from source to destination go via one of the two transhipment nodes.

The transhipment nodes must satisfy the balance equations:

$$X14 + X24 + X34 = X46 + X47 + X48$$

and

$$X15 + X25 + X35 = X56 + X57 + X58.$$

These say that what enters a transhipment node has to be equal to what leaves it.

Transhipment nodes may also be subject to capacity constraints such as

$$X14 + X24 + X34 \leq 600$$

which says that node **T4** can handle no more than 600 units from all sources combined.

ASSIGNMENT PROBLEMS

Assignment problems are a special case of transportation problems where there is a one-to-one allocation of 'sources' to 'destinations'.

An example is the problem of a minicab company with 4 taxis and 4 passengers. Which taxi should collect which passenger in order to minimise costs?

The special restrictions of this problem are

- All allocations are either zero or one
- There will be only n empty cells in the allocation table
  (as compared with m+n-1 in a standard transportation problem)

Special algorithms exist to solve assignment problems. The most common is probably the Hungarian solution method.

An Example

Suppose we have to allocate 4 tasks (1,2,3,4) between 4 people (W,X,Y,Z). The costs are set out in the following table:

	Task			
Person	1	2	3	4
W	8	20	15	17
X	15	16	12	10
Y	22	19	16	30
Z	25	15	12	9

The parenthesised number in each cell shows the cost of each person doing each job.

The assignment problem can be set up as a Linear Programming problem:

Min 8 W1 + 20 W2 + 15 W3 + 17 W4
+ 15 X1 + 16 X2 + 12 X3 + 10 X4
+ 22 Y1 + 19 Y2 + 16 Y3 + 30 Y4
+ 25 Z1 + 15 Z2 + 12 Z3 + 9 Z4

subject to

W1 + W2 + W3 + W4 = 1 (1)
X1 + X2 + X3 + X4 = 1 (2)
Y1 + Y2 + Y3 + Y4 = 1 (3)
Z1 + Z2 + Z3 + Z4 = 1 (4)
W1+ X1 + Y1 + Z1 = 1 (5)
W2 + X2 + Y2 + Z2 = 1 (6)
W3 + X3 + Y3 + Z3 = 1 (7)
W4 + X4 + Y4 + Z4 = 1 (8)

All variables are 0,1 binary integer variables

The Hungarian Solution Method

The Hungarian solution method consists of four steps.

	Task			
Person	1	2	3	4
W	8	20	15	17
X	15	16	12	10
Y	22	19	16	30
Z	25	15	12	9

Step 1: Compute Opportunity Costs for each cell

This step consists of two parts:
a) subtract minimum value from each row
b) subtract minimum value from each column

a) subtract minimum value from each row

	Task			
Person	1	2	3	4
W	0	12	7	9
X	5	6	2	0
Y	6	3	0	14
Z	16	6	3	0

b) **subtract minimum value from each column**

	Task			
Person	1	2	3	4
W	0	9	7	9
X	5	3	2	0
Y	6	0	0	14
Z	16	3	3	0

Step 2: Determine whether optimal assignment can be made

Find the *minimum* number of lines necessary to cross-out all the zero cells in the table. If this is equal to n (the number of people/tasks) then the solution has been found.

	Task			
Person	1	2	3	4
W	0	9	7	9
X	5	3	2	0
Y	6	0	0	14
Z	16	3	3	0

The minimum number of lines necessary to cross through all the zeros is 3 < n so that an optimal allocation has not yet been found.

(Note that there may be more than one way to draw the lines through the zero cells. It does not matter which way you choose as long as there is no alternative way involving fewer lines.)

Step 3: Revise the table

a) find the minimum uncovered cell (2)
b) subtract this from all the uncovered cells
c) add to all the cells at the intersection of 2 lines

	Task			
Person	1	2	3	4
W	0	7	5	9
X	5	1	0	0
Y	8	0	0	16
Z	16	1	1	0

Return to Step 2 to check if the revised allocation is optimal. If not, then repeat Steps 2 and 3 until an optimal allocation is found.

	Task			
Person	1	2	3	4
W	0	7	5	9
X	5	1	0	0
Y	8	0	0	16
Z	16	1	1	0

This time the minimum number of lines necessary to cross through all the zeros is 4 = n so that an optimal allocation has been found.

Step 4: Make the Allocation

Note that only one assignment will be made from each row or column. We use this fact to proceed to making the final allocation as follows:

a) find a row or column with only one zero cell
b) make the assignment corresponding to that zero cell
c) eliminate that row and column from the table
d) continue until all assignments have been made

	Task			
Person	1	2	3	4
W	0	7	5	9
X	5	1	0	0
Y	8	0	0	16
Z	16	1	1	0

- Assign person W to task 1 and eliminate row W and column 1
- Assign person Y to task 2 and eliminate row Y and column 2
- Assign person Z to task 4 and eliminate row Z and column 4
- This leaves final person X assigned to remaining task 3

From the original cost table, we can determine the costs associated with the optimal assignment:

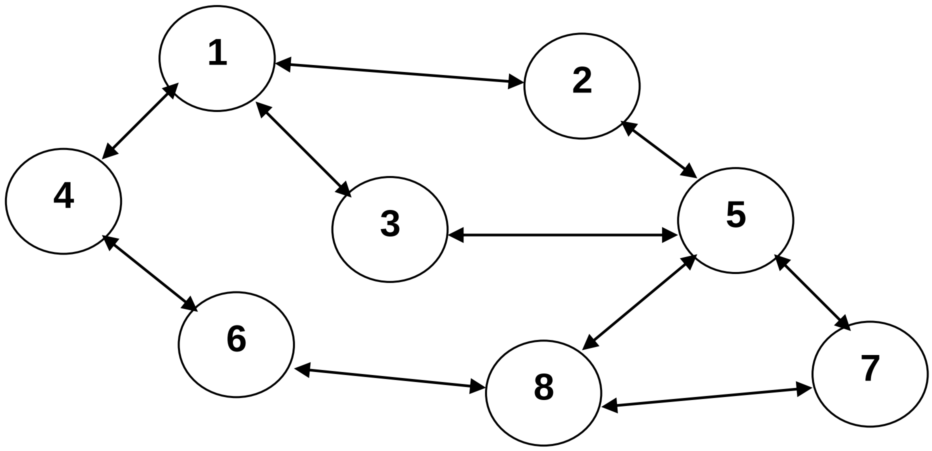
W1 = 8; Y2 = 19; Z4 = 9; X3 = 12; Total cost = 48.

Network Models

The various transportation problems we have examined are particular examples of network models.

- Network models are concerned with the flow of items through a system
- Network models have a simple diagrammatic representation
- Network models can be set up as standard Linear Programming problems and solved using the simplex algorithm
- However, special purpose algorithms exist for these problems that are *simpler* and *more efficient* than simplex

Shortest Route Network Problems

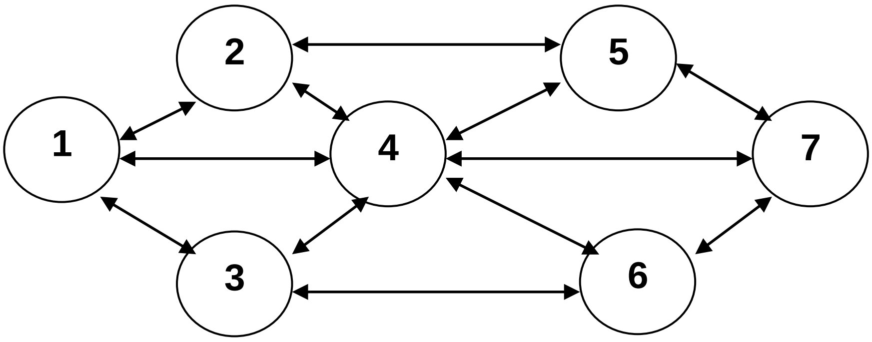


Arrows represent routes and associated with each route is a cost. Note that not all nodes are necessarily directly connected.

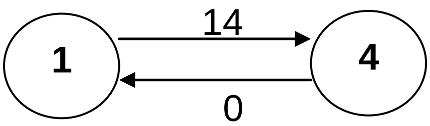
Problem:
What is the shortest (minimum cost route) from Node 1 to Node 7?

Transportation and transhipment problems can be formulated as shortest route problems.

Maximal Flow Network Problems



In this type of problem the arrows represent the flow capacities between each node. These capacities may be different according to the direction of the flow.



The flow capacity X14 from node 1 to node 4 is 14 whereas
The flow capacity X41 from node 4 to node 1 is 0.