Econometrics Lecture 10: Applied Demand Analysis

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1 Introduction

In this lecture we look at the estimation of systems of demand equations. Demand equations were some of the earliest economic relationships to be analysed using statistical techniques. Demand analysis is an example of an applied area where there is a very well developed economic theory, consumer theory, that implies several parameter restrictions. Various econometric models have been developed in which it is possible to test at least some of these restrictions. From an econometric point of view, these models are interesting in that they involve nonlinearity and complete system estimation methods.

Demand analysis is a large subject and several books have been written in this area. In this lecture we will only be able to touch on some of the issues. For those who are interested in reading further, the following are recommended: the books by Deaton and Muellbauer (1980b), Phlips (1983) and Thomas (1987) as well as the survey article by Deaton (1986).

2 The Economic Theory

The neo-classical theory of consumer behaviour assumes that each consumer chooses demand so as to maximise utility subject to a budget constraint. Formally the problem can be written as

$$\max_{q} u(\mathbf{q}) \quad \text{subject to } \mathbf{p}'\mathbf{q} \le m \tag{2.1}$$

where $u(\mathbf{q})$ is the utility function, $\mathbf{q}$ is an $n \times 1$ vector of quantities demanded of each of the n commodities, $\mathbf{p}$ is an $n \times 1$ vector of prices, and m is total expenditure. The solution to this problem is given by the *indirect utility function*

$$v(\mathbf{p}, m)\,.$$

Generally, it is more convenient to work with the *dual* of this maximisation problem which is the problem of minimising expenditure subject to a given level of utility, u. Formally this is written as

$$\min_q \mathbf{p}'\mathbf{q} \quad \text{subject to } u(\mathbf{q}) \geq u\,. \tag{2.2}$$

The solution to this problem is the *cost function*

$$c(\mathbf{p}, u)$$

which is assumed to be *concave* in prices. The partial derivatives of the cost function with respect to prices

$$\frac{\partial c(\mathbf{p}, u)}{\partial p_i} \equiv h_i(\mathbf{p}, u) = q_i$$

are the *compensated* or *Hicksian* demand functions. These give the demand for good i, q_i, when the consumer is compensated so as to maintain a constant utility level u.

The second derivatives

$$\frac{\partial^2 c(\mathbf{p}, u)}{\partial p_i \partial p_j} = \frac{\partial h_i(\mathbf{p}, u)}{\partial p_j} = s_{ij}$$

form a matrix

$$\mathbf{S} = \begin{bmatrix} s_{11} & \cdots & s_{1n} \\ \vdots & \ddots & \vdots \\ s_{n1} & \cdots & s_{nn} \end{bmatrix} = \frac{\partial \mathbf{h}(\mathbf{p}, u)}{\partial \mathbf{p}'} = \frac{\partial^2 c(\mathbf{p}, u)}{\partial \mathbf{p} \partial \mathbf{p}'} \tag{2.3}$$

which is known as the *substitution matrix* or the *Slutsky* matrix. It follows from the concavity of the cost function that this matrix has to be *symmetric* and *negative semi-definite*. In particular, the diagonal terms of this matrix which are the own-price substitution effects should always be negative. This means, that *as long as a consumer is compensated so as to maintain a constant utility*, then a rise in the price of a good should lead to a fall in demand.

The *Marshallian* demand function

$$q_i = \phi_i(\mathbf{p}, m)$$

gives demand as a function of prices and expenditure. This can be derived from the cost function since the solution to the maximisation problem (2.1) and the minimisation problem (2.2) is the same so that

$$q_i = h_i(\mathbf{p}, u) = h_i(\mathbf{p}, v(\mathbf{p}, m)) = \phi_i(\mathbf{p}, m)$$

where the indirect utility function has been substituted into the cost function to derive an expression for the demand function in terms of observables.

Differentiating this expression with respect to prices gives

$$\begin{aligned} s_{ij} &\equiv \frac{\partial h_i(\mathbf{p},u)}{\partial p_j} = \frac{\partial \phi_i(\mathbf{p},m)}{\partial p_j} + \frac{\partial \phi_i(\mathbf{p},m)}{\partial m}\frac{\partial m}{\partial p_j} \\ &= \frac{\partial \phi_i(\mathbf{p},m)}{\partial p_j} + \frac{\partial \phi_i(\mathbf{p},m)}{\partial m} q_j \end{aligned} \tag{2.4}$$

and the second line of this can be rearranged as

$$\frac{\partial \phi_i(\mathbf{p},m)}{\partial p_j} = s_{ij} - \frac{\partial \phi_i(\mathbf{p},m)}{\partial m} q_j \,. \tag{2.5}$$

This equation is called the *Slutsky equation.* The first term is the *substitution effect* of a change in price and the second is the *income effect.* For $i = j$ we have seen that the substitution effect must be negative. The income effect may be positive or negative. If it is positive then the good is a *normal good* and the net effect of a price increase is to reduce demand. If the income effect is negative, then the good is an *inferior good.* In this case it is possible but unlikely for the income effect to outweigh the substitution effect so that the net effect of a price increase is to increase demand. A good with this peculiar property is known as a *Giffen good.*

2.1 Properties of Demand Functions

Consumer demand theory implies that demand functions have four properties: *additivity*, *homogeneity*, *negativity*, and *symmetry.*

Additivity is the requirement that total expenditure on all goods exhausts the budget so that

$$m = \sum_{i=1}^{n} p_i q_i = \mathbf{p}'\mathbf{q}. \tag{2.6}$$

It can be shown that both the Hicksian and Marshallian demand functions must be *homogeneous* of degree zero in prices and total expenditure, so that if all prices and expenditure change proportionally, then there is no change in demand and

$$q_i = \phi_i(\mathbf{p},m) = \phi_i(k\mathbf{p},km) \tag{2.7}$$

for any k.

It follows from the Slutsky equation (2.5) that

$$s_{ii} = \frac{\partial h_i(\mathbf{p},u)}{\partial p_i} = \frac{\partial \phi_i(\mathbf{p},m)}{\partial p_i} + \frac{\partial \phi_i(\mathbf{p},m)}{\partial m} q_i < 0$$

This is the *negativity* restriction for the compensated demand functions that follows from the negative definiteness of the Slutsky matrix **S**. This restriction of *negativity* must also apply to all linear combinations of commodities.

Finally the cross-price derivatives of the compensated demand functions must be *symmetric* so that, allowing for compensation to maintain utility constant, the effect of a change in the price of good i on the demand for good j is the same as the effect of a change in the price of good j on the demand for good i. Algebraically, this restriction is given by

$$\frac{\partial h_i}{\partial p_j} = \frac{\partial h_j}{\partial p_i}$$

or

$$\frac{\partial q_i}{\partial p_j} + q_j \frac{\partial q_i}{\partial m} = \frac{\partial q_j}{\partial p_i} + q_i \frac{\partial q_j}{\partial m} \,. \tag{2.8}$$

3 Single Equation Estimates

A single equation from a demand system takes the general form

$$q_i = \phi_i(p_1, \cdots, p_n, m) \,. \tag{3.1}$$

Only two of the four conditions derived from economic theory are relevant to single equations: the homogeneity restrictions and the negativity restrictions. Demand theory does not specify any particular functional form for the utility function and consequently neither for the demand functions (3.1). In specifying an equation to estimate, the applied econometrician needs to make some assumption about the functional form to be used which implies a corresponding assumed functional form for the utility function. The earliest applied work used linear functional forms. These have the advantage of adding-up easily. However, the linear function form implies that price elasticities are not constant but depend on the absolute level of prices. Thus logarithmic functional forms, in which elasticties are constant have been more popular.

One important issue is that of simultaneity. In general it is assumed that prices are fixed to the consumer, and so can be treated as exogenous. However, total expenditure m includes the expenditure on commodity i, $p_i q_i$ which involves the dependent variable, q_i. Thus expenditure is in principle an endogenous variable. Some applied work does take this into account and uses an instrument for expenditure in the equation (3.1). On the other hand, the majority of empirical work takes m as exogenous, arguing that this is a good enough approximation, as long as q_i is small as a component of m. This justifies the use of *OLS* estimation.

Another issue is that of dynamics. The equation (3.1) is static, relating current consumption to current prices and current total expenditure. In practice, adjusting consumption in reponse to a change in prices might be expected to take time, both because of habit persistence and the lumpiness of consumption. Where time series data is available, then it is possible to estimate dynamic demand functions. However, much work uses cross-sectional data from household budget surveys.

A seminal applied study for the UK was that of Stone (1954a). Stone started from the logarithmic model

$$\log q_i = \alpha_i + e_i \log m + \sum_{j=1}^{n} e_{ij} \log p_j \tag{3.2}$$

where e_i is the total expenditure elasticity and e_{ij} are the cross-price elasticities of the jth price on demand for good i. This model has $n+2$ parameters. Stone wished to use this model to estimate demand for 48 categories of food consumption with *UK* data over the period 1920-38 (19 observations). Clearly estimation is impossible without imposing some prior restrictions. Setting some of the price elasticities e_{ij} to zero imposes both no substitution effect but also no income effect. Whereas the former might be expected to be zero for 'unrelated goods', the latter would not be exxpected to be zero. Stone solved this problem by using consumer theory to decompose the elasticities.

From the Slutsky equation (2.5) it follows that

$$\frac{\partial \phi_i(\mathbf{p},m)}{\partial p_j}\frac{p_j}{q_j} = \frac{\partial h_i(\mathbf{p},u)}{\partial p_j}\frac{p_j}{q_j} - \frac{\partial \phi_i(\mathbf{p},m)}{\partial m}\frac{m}{q_j}\frac{p_j q_j}{m}$$

or

$$e_{ij} = e^*_{ij} - e_i w_j \tag{3.3}$$

where

$$e^*_{ij} = \frac{\partial h_i(\mathbf{p},u)}{\partial p_j}\frac{p_j}{q_j}$$

is the *compensated cross-price elasticity* and w_j is the budget share defined by

$$w_j = \frac{p_j q_j}{m}\,. \tag{3.4}$$

Substituting (3.3) into (3.2) gives

$$\log q_i = \alpha_i + e_i(\log m - \sum_{j=1}^{n} w_j \log p_j) + \sum_{j=1}^{n} e^*_{ij} \log p_j$$

but the term $\sum_{j=1}^{n} w_j \log p_j$ can be thought of as the logarithm of a general price index P so that the equation can be rewritten as

$$\log q_i = \alpha_i + e_i \log(m/P) + \sum_{j=1}^{n} e^*_{ij} \log p_j\,. \tag{3.5}$$

This gives demand in terms of real expenditure and compensated prices. Since the compensated price elasticities comprise pure substitution effects, it is reasonable to allow that many of these will be zero for unrelated goods, so that the summation in (3.5) can be taken over the subset of related goods only.

Finally, homogeneity of (3.5) implies that

$$\sum_j e^*_{ij} = 0\,.$$

This can be imposed by deflating all prices by the general price index P to give

$$\log q_i = \alpha_i + e_i \log(m/P) + \sum_j e^*_{ij} \log(p_j/P)\,. \tag{3.6}$$

This equation was the basis for Stone's analysis. Sample results from Stone (1954a) are given in a table at the end of this handout, taken from Deaton and Muellbauer (1980).

An advantage of the specification (3.6) is that it estimates income and substitution effects with constant elasticities. It imposes homogeneity but allows the possibility of testing negativity restrictions on own-price elasticities. A drawback is that the equation (3.6) cannot be derived from a reasonable utility function except in the trivial case where income and own-price elasticities are unity and cross-elasticities are zero. Thus it is theoretically a little unsatisfactory. Despite this, it was used with considerable effect by Stone to predict the effect on consumption of the ending of rationing after the Second World War.

4 Demand Systems

Estimating single equations from a demand system is simple and comparatively cheap. However, it ignores the important economic restrictions of *additivity* and *symmetry* which involve cross-equation restrictions that cannot be imposed equation by equation. As a result, single equation parameter estimates will be less efficient than those that come from estimation of the complete system. On the other hand, it is possible to estimate much larger systems with single equation methods than with system methods because the latter will involve more parameters.

For these reasons, there have been several attempts to estimate complete systems of demand equations, imposing all the restrictions of economic theory. One problem is finding an appropriate specification that makes it possible to impose the adding-up restriction. We look at three different demand systems that have been estimated: the *Linear Expenditure System*, the *Rotterdam model* and the *Almost Ideal Demand System*.

5 The Linear Expenditure System

The *Linear Expenditure System* or *LES* can be derived by imposing the restrictions of adding-up, homogeneity and symmetry on the linear functional form

$$p_i q_i = \beta_i m + \sum_{j=1}^{n} p_j \alpha_{ij} \, .$$

The resulting model is defined by the equation

$$p_i q_i = p_i \gamma_i + \beta_i (m - \sum_{j=1}^{n} p_j \gamma_j) + u_i \tag{5.1}$$

where $\sum_{i=1}^{m} \beta_i = 1$. This model was used by Stone (1954b). It can be interpreted as a hypothesis that expenditure on good i can be decomposed into two components. The first is a certain minimum expenditure on the good given by the quantity $p_i \gamma_i$. This corresponds to a subsistence level of expenditure that always takes place regardless of income. The second component is a function of that part of income m above the subsistence income $\sum_{j=1}^{n} p_j \gamma_j$ used to purchase the subsistence expenditure on all goods. This super-numerary income is allocated between the goods in the fixed proportions β_i. Note that this interpretation requires that the coefficients β_i are positive which is not actually required in the model.

The *Linear Expenditure System* implies a utility function of the form

$$u(\mathbf{q}) = \prod_{j=1}^{n} (q_j - \gamma_j)^{\beta_j} \, .$$

This is known as the *Stone-Geary utility function*. Since the model has already imposed the restrictions of economic theory, these cannot be tested within the model. There are $2n$ parameters in the model, only $2n - 1$ of which can be determined independently. This is far fewer than the $\frac{1}{2}n(n+1) - 1$ that are permitted by economic theory. This is because the assumption of a linear functional form is very restrictive, and imposes restrictions over and above those that derive from economic theory. In the *LES* model no two goods can be complements; every good has to be a substitute for every other good. On the other hand, the small number of parameters does mean that quite large demand systems, with more than 40 goods, can be estimated.

The model is linear in the variables but not in the parameters β_i and γ_i. Thus it requires a nonlinear estimation method. If the γ parameters are known, then (5.1) is linear in the β parameters. Conversely, if the β parameters are known, then the model is linear in the γ's. Stone (1954b) estimated the model by using this fact to iterate between the β's and γ's. However, with modern computers, the nonlinear problem can be solved directly using maximum likelihood methods.

One technical problem arises from the identity that

$$\sum_{i=1}^{m} p_i q_i = m\,. \tag{5.2}$$

Summing the equations (5.1) gives

$$\sum_{i=1}^{m} p_i q_i = \sum_{i=1}^{m} p_i \gamma_i + m \sum_{i=1}^{m} \beta_i - \sum_{i=1}^{m} \beta_i \sum_{j=1}^{n} p_j \gamma_j + \sum_{i=1}^{m} u_i$$

but $\sum_{i=1}^{m} \beta_i = 1$ which implies that $\sum_{i=1}^{m} u_i = 0$ so that the covariance matrix

$$\mathrm{var}(\mathbf{u}) = \mathbf{\Sigma}$$

will be *singular*. This result follows from the fact that the system adds up so that the errors are not linearly independent but sum to zero. Luckily, this problem can easily be solved by dropping one of the equations and estimating the other $n-1$ by maximum likelihood. The parameters for the missing equation can be recovered from those of the other equations and the adding-up identity (5.2). It can be proved that it makes no difference which of the equations is dropped from the analysis.

The Linear Expenditure System was the first complete demand system to be estimated. It has the advantages of imposing all the restrictions of economic theory, having a small number of parameters to be estimated, and coming from a well-defined utility function. On the other hand, this utility function is very restrictive, and rules out complementary goods. It also imposes approximate proportionality of price and expenditure elasticities, which is a restriction not demanded by economic theory. Furthermore, the economic restrictions imposed on the model cannot be relaxed. Thus it is not possible to test the restrictions implied by consumer demand theory against the data.

6 The Rotterdam Model

The Rotterdam model was first developed by Theil (1965) and Barten (1966). It can be derived from first differencing the equation (3.6) from the Stone (1954a) model to give

$$\Delta \log q_i = e_i(\Delta \log m - \sum_{j=1}^{n} w_j \Delta \log p_j) + \sum_j e_{ij}^* \Delta \log p_j\,.$$

Multiplying this equation by the budget share w_i defined by (3.4), it can be rewritten in the form

$$w_i \Delta \log q_i = b_i \Delta \log \overline{m} + \sum_{j=1}^{n} c_{ij} \Delta \log p_j \tag{6.1}$$

where

$$\Delta \log \overline{m} \equiv \Delta \log m - \sum_{j=1}^{n} w_j \Delta \log p_j = \sum_{j=1}^{n} w_j \Delta \log q_j$$

and the parameters b_i and c_{ij} are defined by

$$b_i = w_i e_i \quad \text{and} \quad c_{ij} = w_i e^*_{ij} \, .$$

This transformation makes it very easy to specify the economic restrictions of *additivity*, *homogeneity* and *symmetry* in terms of the parameters of the model.

The adding-up restrictions on this model imply that

$$\sum_{j=1}^{n} b_j = 1 \quad \text{and} \quad \sum_{k=1}^{n} c_{kj} = 0 \, .$$

These restrictions will hold in the model and cannot be tested. Homogeneity implies that

$$\sum_{k=1}^{n} c_{ik} = 0 \quad , \quad i = 1, \cdots, n$$

and symmetry implies that

$$c_{ij} = c_{ji}, \quad , \quad \forall i, j \, .$$

These two sets of restrictions can be tested in the model.

The Rotterdam model was estimated by Barten (1969) using Dutch data for the post-war period on 16 commodity groups. The restrictions of homogeneity were strongly rejected. Symmetry was also rejected but less convincingly.

The Rotterdam model has been criticised on theoretical grounds because the parameters b_i and c_{ij} in (6.1) are only strictly constant when all total expenditure elasticities are equal to one, all own-price elasticities are equal to minus one and all cross-price elasticities are equal to zero. This is because the differenced form of demand function (6.1) is only consistent with the demand function in levels under these conditions. In practice this criticism may not be too serious, as long as b_i and c_{ij} are close to being constant. It should always be borne in mind, however, that the Rotterdam model is just an approximation to demand theory and may not work well in all circumstances.

The Rotterdam model cannot be derived from a well-behaved utitility function, unlike the *LES* system. On the other hand, it is less restrictive than the *LES* and allows the economic restrictions to be tested.

7 The Almost Ideal Demand System (AIDS)

The *Almost Ideal Demand System* or *AIDS* was developed by Deaton and Muellbauer (1980a). This model is defined by the equation

$$w_i = \alpha_i + \sum_{j=1}^{n} \gamma_{ij} \log p_j + \beta_i \log(m/P) + u_i \tag{7.1}$$

where w_i is again the budget share and P is a price index defined by

$$\log P = \alpha_0 + \sum_{j=1}^{n} \alpha_j \log p_j + \frac{1}{2} \sum_{j=1}^{n} \sum_{k=1}^{n} \gamma_{jk} \log p_j \log p_k \,. \tag{7.2}$$

Adding-up requires that

$$\sum_{k=1}^{n} \alpha_k = 1 \quad , \quad \sum_{k=1}^{n} \beta_k = 0 \quad , \quad \sum_{k=1}^{n} \gamma_{kj} = 0 \,,$$

while *homogeneity* requires

$$\sum_{k=1}^{n} \gamma_{ik} = 0$$

and *symmetry* requires

$$\gamma_{ij} = \gamma_{ji} \,.$$

The adding-up restrictions will automatically be satisfied and so are not testable. However, both the homogeneity and symmetry restrictions can be tested. The *negativity* conditions can also be tested by the negative-definiteness of the matrix formed from the elements

$$s_{ij} = \gamma_{ij} + \beta_i \beta_j \log(m/P) - w_i \delta_{ij} + w_i w_j$$

where δ_{ij} is the Kronecker delta taking the value 1 when $i = j$ and zero otherwise.

The β_i parameters will be negative for necessities and positive for luxury goods. The γ_{ij} parameters measure the change in the ith budget share following a proportional change to p_j where real income as measured by m/P is held constant.

The *Almost Ideal Demand System* has the advantage of being nearly linear. Apart from the expression for P given by equation (7.2), the system can be estimated equation by equation. P can be proxied by any appropriately defined price index which can be calculated before estimation in which case equation (7.1) can be directly estimated by *OLS*. This is one advantage of *AIDS* over the

Rotterdam model. The other advantage is that *AIDS* can be derived from a well-behaved utility function of the general *PIGLOG* class with cost function defined by

$$\log c(\mathbf{p}, u) = u \log b(\mathbf{p}) + (1 - u) \log a(\mathbf{p})$$

where $a(\mathbf{p})$ and $b(\mathbf{p})$ are linear homogeneous functions. In the particular case of the *AIDS* model, these functions are given by

$$\log a(\mathbf{p}) = \log(P) = \alpha_0 + \sum_{j=1}^{n} \alpha_j \log p_j + \frac{1}{2} \sum_{j=1}^{n} \sum_{k=1}^{n} \gamma_{jk} \log p_j \log p_k$$

and

$$\log b(\mathbf{p}) = \log a(\mathbf{p}) + \beta_0 \prod_{j=1}^{n} p_j^{\beta_j} .$$

Deaton and Muellbauer (1980a) report estimates of this model for the UK from 1954-74. The restriction of homogeneity was rejected for food, clothing, housing and transport. The restrictions of symmetry were also rejected although the evidence for this is less clear-cut. Own-price elasticities were generally found to be less than one in absolute value. These results are similar to those obtained by Barten (1969) on Dutch data using the Rotterdam model.

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