Financial Econometrics
Lecture 4: Extending GARCH models and Stochastic Volatility models

Richard G. Pierse

1 Introduction

The *ARCH* model of Engle (1982) and its *GARCH* generalisation by Bollerslev (1986) can be extended in many different ways. *IGARCH* models extend the framework to allow volatility to have persistence, whereas *ARCH-M* models allow a correlation between the mean and volatility of a series and *EGARCH* models allow an asymmetry between the effect of positive and negative shocks. Stochastic volatility models take a difference approach by treating volatility as an unobservable stochastic variable.

2 Integrated *(G)ARCH* processes

In some empirical applications the stationarity condition that

$$\sum_{i=1}^{q} \alpha_i + \sum_{j=1}^{p} \beta_j < 1$$

is not met. In this case, the conditional variance will exhibit persistence to shocks. A special case is where

$$\sum_{i=1}^{q} \alpha_i + \sum_{j=1}^{p} \beta_j = 1$$

which is known as an *Integrated GARCH* or *I-GARCH* process. In this model, first considered by Engle and Bollerslev (1986), the *ARMA* process for u_t^2 has a *unit root*. However, although the process is not covariance stationary since the unconditional variance of u_t is infinite, Nelson (1990) showed that the *IGARCH* model may still be strictly stationary, so that the effect of shocks to volatility will eventually die out. In the *IGARCH(1,1)* case, a necessary condition for this is that

$$\mathrm{E}(\log(\beta_1 + \alpha_1 u_{t-j}^2 \sigma_{t-j}^2)) < 0 \quad j = 1, \cdots, t.$$

The RiskMetrics methodology developed at J.P. Morgan uses a special case of the *IGARCH(1,1)* model to explain the daily log return of a portfolio.

3 *(G)ARCH-M* processes

In the standard *(G)ARCH* model, the time-varying volatility of u_t has no effect on the conditional mean of the process, which is given by

$$\mathrm{E}(y_t | y_{t-1}) = \mathbf{x}_t' \beta.$$

In many financial applications, this might be unrealistic. For example, one might expect that periods of high volatility in returns might correspond with periods when expected returns are high. Engle, Lilien and Robins (1987) consider the *ARCH-in-Mean* or *ARCH-M* model in which the conditional variance σ_t^2 is allowed directly to influence the mean of the process. The model is defined by:

$$y_t = \mathbf{x}_t' \beta + \gamma \sigma_t^2 + u_t \quad , \quad t = 1, \cdots, T \tag{3.1}$$

where σ_t^2 follows the *ARCH* process

$$\sigma_t^2 = \alpha_0 + \alpha_1 u_{t-1}^2 + \alpha_2 u_{t-2}^2 + \cdots + \alpha_q u_{t-q}^2 \tag{3.2}$$

or the *GARCH* process

$$\sigma_t^2 = \alpha_0 + \alpha_1 u_{t-1}^2 + \alpha_2 u_{t-2}^2 + \cdots + \alpha_q u_{t-q}^2 + \beta_1 \sigma_{t-1}^2 + \beta_2 \sigma_{t-2}^2 + \cdots + \beta_p \sigma_{t-p}^2. \tag{3.3}$$

In this model, the conditional mean

$$\mathrm{E}(y_t | y_{t-1}) = \mathbf{x}_t' \beta + \gamma \sigma_t^2$$

in linearly related to the conditional variance with coefficient γ.

4 *EGARCH* processes

In the standard *(G)ARCH* model, volatility is symmetric with respect to positive and negative shocks. In actual financial data, it is often observed that there is an asymmetry between the volatility associated with positive and negative shocks. A model which allows for such asymmetry is the *exponential GARCH* or *EGARCH* model of Nelson (1991), with conditional variance σ_t^2 defined by

$$\log \sigma_t^2 = \alpha_0 + \sum_{i=1}^{p} \alpha_i \frac{u_{t-i}}{\sigma_{t-i}} + \sum_{i=1}^{p} \gamma_i \frac{|u_{t-i}| - \mathrm{E}(|u_{t-i}|)}{\sigma_{t-i}} + \sum_{j=1}^{p} \beta_j \log \sigma_{t-j}^2 \tag{4.1}$$

where $|u_{t-i}|$ is the absolute value of u_{t-i}. In this model, positive and negative values of u_{t-i} have different effects on volatility. A positive shock in period $t-i$ has effect $\alpha_i + \gamma_i$ whereas a negative shock has effect $\gamma_i - \alpha_i$. The logarithmic formulation of this model means that the conditional variance σ_t^2 cannot be negative, whatever values are taken by the coefficients.

5 Stochastic Volatility Models

(G)ARCH models represent one, very important, approach to modelling processes with time-varying conditional volatility. An alternative approach is provided by stochastic volatility models. The simplest stochastic volatility model, introduced by Taylor (1986), is defined by the equations

$$y_t = \sigma_t u_t \tag{5.1}$$

where the log-volatility, $\log \sigma_t$, follows the autoregressive process

$$\log \sigma_t = \phi_0 + \phi_1 \log \sigma_{t-1} + \omega \varepsilon_t. \tag{5.2}$$

u_t and ε_t are independent white noise processes with unit variance. One important difference between this stochastic volatility model and the family of *(G)ARCH* models is that here the conditional variance σ_t depends on an additional noise process ε_t and so is itself an unobservable variable. This makes stochastic volatility models much more difficult to estimate than *GARCH* models as the likelihood function cannot be written down directly, and alternative estimation techniques must be used. A survey of these techniques is presented in Shephard (1996). We consider only the simplest case where the conditional mean of y_t, $y_t|y_{t-1}$, is zero. In this model, it is possible to transform the model into a linear form.

To convert the simple stochastic model to a linear form, we would like to take logarithms of (5.1) but, since y_t can be negative, we must first take absolute values. Then

$$\log |y_t| = \log \sigma_t + E \log |u_t| + (\log |u_t| - E \log |u_t|). \tag{5.3}$$

where the term in brackets is treated as a normally distributed zero mean disturbance. Equations (5.3) and (5.2) together form what is called a linear state-space form. In such a state-space form model, it is possible to compute the likelihood recursively, using an algorithm called the Kalman filter. (See Harvey *et al.* (1994) for details.) The estimation can be perfomed using the *STAMP* computer software package of Koopman *et al.* (1995).

Stochastic volatility models are quite closely related to *EGARCH* models. Both models use a logarithmic formulation of the conditional volatility so that no further conditions are necessary to ensure that σ_t^2 is positive. Both models imply excess kurtosis, since in the stochastic volatity model

$$\kappa(y_t) = 3\exp(\mathrm{var}(\log\sigma_t)) = 3\exp\frac{\omega^2}{1-\phi_1^2} > 3,$$

although the excess kurtosis here will in general be larger than in *GARCH* models and has no upper limit. This makes the stochastic volatility approach attractive for dealing with processes with very large kurtosis. However, the difficulty of estimation of these models compared with *GARCH* models has limited their use.

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