Financial Econometrics
Lecture 5: The Capital Asset Pricing Model

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1 Introduction

So far in this module, the models for stock returns we have considered have been purely statistical. In this lecture we look at an economic model from finance theory that explains the expected return on a financial asset as a function of its relative risk and on the excess return of the market portfolio.. This is the famous Capital Asset Pricing Model (*CAPM*) developed by Sharpe (1964) (who won a Nobel prize in 1990 for his work), Lintner (1965) and Mossin (1966).

2 Derivation of the *CAPM*

The Capital Asset Pricing Model (*CAPM*) of Sharpe (1964), Lintner (1965) and Mossin (1966) is the first and most widely used model in asset pricing. A portfolio comprises n risky assets and one risk-free asset. Let x_i be the proportion of the portfolio in asset i so that $1 - \sum x_i$ is the proportion in the risk-free asset. The return on the portfolio, r_p, is then defined by

$$r_p = \mathbf{x}'\mathbf{r} + r_f(1 - \mathbf{x}'\iota) \tag{2.1}$$

where $\mathbf{r}$ is an $n \times 1$ vector of returns on the n risky assets, r_f is the return on the risk-free asset and $\mathbf{x}$ is an $n \times 1$ vector of the portfolio proportions with typical element x_i. Since the return on the risky assets is uncertain, so is the return on the portfolio, and it will have expected value

$$\mathrm{E}(r_p) = \mathbf{x}'\,\mathrm{E}(\mathbf{r}) + r_f(1 - \mathbf{x}'\iota) \tag{2.2}$$

and variance

$$\mathrm{var}(r_p) = \sigma_p^2 = \mathbf{x}'\,\mathrm{var}(\mathbf{r})\mathbf{x}$$

where $\mathrm{var}(\mathbf{r})$ is an $n \times n$ variance covariance matrix.

Investors are assumed to choose their portfolio to minimise its risk, defined by the square root of variance, σ_p, for a given rate of expected return. Formally they must choose $\mathbf{x}$ to solve the Lagrangian problem

$$\min_{\mathbf{x}} c = \operatorname{var}(r_p)^{\frac{1}{2}} + \lambda(\mathrm{E}(r_p) - \mathbf{x}' \mathrm{E}(\mathbf{r}) - r_f(1 - \mathbf{x}'\iota)) \tag{2.3}$$

where λ is a Lagrangian multiplier. First order conditions for this constrained minimisation problem are

$$\frac{\partial c}{\partial \mathbf{x}} = \sigma_p^{-1} \operatorname{var}(\mathbf{r})\mathbf{x} - \lambda(\mathrm{E}(\mathbf{r}) - r_f\iota) = \mathbf{0} \tag{2.4}$$

and

$$\frac{\partial c}{\partial \lambda} = \mathrm{E}(r_p) - \mathbf{x}' \mathrm{E}(\mathbf{r}) - r_f(1 - \mathbf{x}'\iota) = 0. \tag{2.5}$$

Premultiplying (2.4) by $\mathbf{x}'$ gives

$$\begin{aligned} & \sigma_p^{-1}\mathbf{x}' \operatorname{var}(\mathbf{r})\mathbf{x} - \lambda(\mathbf{x}' \mathrm{E}(\mathbf{r}) - r_f\mathbf{x}'\iota) \\ = \; & \sigma_p - \lambda(\mathbf{x}' \mathrm{E}(\mathbf{r}) - r_f\mathbf{x}'\iota) = \mathbf{0}. \end{aligned}$$

Now consider a special portfolio where $\mathbf{x}'\iota = 1$ so that the risk-free asset is excluded. This portfolio will be called the *market portfolio*. The return on the market portfolio, denoted r_m, is given by

$$r_m = \mathbf{x}'\mathbf{r}$$

with expected value

$$\mathrm{E}(r_m) = \mathbf{x}' \mathrm{E}(\mathbf{r})$$

and risk, σ_m, given by

$$\sigma_m = \lambda(\mathrm{E}(r_m) - r_f) \tag{2.6}$$

Substituting (2.6) into (2.4) gives

$$\begin{aligned} \mathrm{E}(\mathbf{r}) &= r_f + \frac{1}{\lambda}\sigma_m^{-1} \operatorname{var}(\mathbf{r})\mathbf{x} \\ &= r_f + (\mathrm{E}(r_m) - r_f)\sigma_m^{-2} \operatorname{var}(\mathbf{r})\mathbf{x}. \end{aligned}$$

Note that $\operatorname{var}(\mathbf{r})\mathbf{x} = \operatorname{cov}(\mathbf{r}, r_m)$ since, by definition,

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$$\begin{aligned} \operatorname{var}(\mathbf{r})\mathbf{x} &= \mathrm{E}(\mathbf{r} - \mathrm{E}(\mathbf{r}))(\mathbf{r} - \mathrm{E}(\mathbf{r}))'\mathbf{x} \\ &= \mathrm{E}(\mathbf{r} - \mathrm{E}(\mathbf{r}))(\mathbf{x}'\mathbf{r} - \mathbf{x}' \mathrm{E}(\mathbf{r}))' \\ &= \mathrm{E}(\mathbf{r} - \mathrm{E}(\mathbf{r}))(r_m - \mathrm{E}(r_m))' \\ &= \operatorname{cov}(\mathbf{r}, r_m). \end{aligned}$$

Thus we can write

$$\begin{aligned}\mathrm{E}(\mathbf{r}) &= r_f + (\mathrm{E}(r_m) - r_f)\frac{\mathrm{cov}(\mathbf{r}, r_m)}{\mathrm{var}(r_m)} \\ &= r_f + (\mathrm{E}(r_m) - r_f)\boldsymbol{\beta}\end{aligned}$$

where $\boldsymbol{\beta} = \mathrm{cov}(\mathbf{r}, r_m)/\,\mathrm{var}(r_m)$ can be considered as an $n \times 1$ vector of parameters. For an individual asset in the portfolio, the expected return, $\mathrm{E}(r_i)$, is given by

$$\mathrm{E}(r_i) = r_f + (\mathrm{E}(r_m) - r_f)\beta_i \tag{2.7}$$

which is the familiar formulation of the *CAPM* model, where the coefficient.

$$\beta_i = \frac{\mathrm{cov}(r_i, r_m)}{\mathrm{var}(r_m)}$$

represents the relative riskiness of the ith asset in the market portfolio. The *CAPM* equation says that the expected excess return of any asset $\mathrm{E}(r_i) - r_f$ is proportional to the expected excess return of the market portfolio $\mathrm{E}(r_m) - r_f$, where the constant of proportionality is the relative riskiness of the asset, β_i. Note that the market portfolio has a β_i equal to unity.Assets for which $\beta_i > 1$ are described as aggressive stocks and assets for which $\beta_i < 1$ are described as defensive stocks. The Lagrange multiplier λ in (2.6) is sometimes known as the *market price of risk*. It represents the slope of a straight line in expected return/risk space known as the *capital market line*. All portfolios held by investors will lie along this line.

The *CAPM* has some important implications. Firstly, it implies that all investors will hold their risky assets in the same relative proportions, $x_i/(\iota'\mathbf{x})$, regardless of their preferences for risk versus return. These proportions correspond to those of the market portfolio. $\iota'\mathbf{x}$ represents the proportion of an individual's portfolio in risky assets. Individuals with different preferences for risk will choose to hold different proportions of risky to risk-free assets.

3 The zero-beta *CAPM*

Sometimes it is not reasonable to assume the existence of a risk-free asset. For example, in a world of inflation and in the absence of index-linked bonds, the real return of any asset will be uncertain. Also, it is generally not possible for investors to borrow unlimited amounts at a riskless rate. The *CAPM* model can be extended to the case where there is no risk-free asset. Instead we assume the existence of a portfolio, with expected return $\mathrm{E}(r_0)$, which is uncorrelated with the market portfolio so that

$$\frac{\mathrm{cov}(r_0, r_m)}{\mathrm{var}(r_m)} = \beta_0 = 0$$

In this model it is possible to derive a version of the *CAPM*, due to Black (1972), called the *zero-beta CAPM* in which it can be shown that

$$\mathrm{E}(r_i) = \mathrm{E}(r_0) + (\mathrm{E}(r_m) - \mathrm{E}(r_0))\beta_i. \tag{3.1}$$

In this model, it is no longer true that all investors hold their risky assets in the same relative proportions. This is because the combination of market portfolio and zero-beta portfolio is not unique. However, every investor can achieve his own optimal portfolio by combining any mean-variance efficient portfolio m with its corresponding zero-beta portfolio.

4 Estimating the *CAPM*

We now turn to consider the estimation of the *CAPM*. The standard *Sharpe-Lintner* form of the *CAPM* where a risk-free asset is assumed to exist is straightforward to estimate. The *Black* or *zero-beta* form of the *CAPM* is more complicated because the zero-beta portfolio is not directly observed and implies a nonlinear parameter restriction. In both cases, in order to estimate the parameters we need time-series data and, since the *CAPM* is a single-period model, this means making extra assumptions about the time-series behaviour of the model. The simplest assumption is that returns are independently and identically distributed through time and jointly multivariate normal.

4.1 Estimating the Sharpe-Lintner *CAPM*

First, consider the estimation of the standard Sharpe-Lintner form of the *CAPM* defined by (2.7). We will assume that returns are independently and identically distributed through time and jointly multivariate normal. Then we can write

$$y_{it} = \alpha_i + \beta_i x_t + u_{it}, \qquad i = 1, \cdots, n, \quad t = 1, \cdots, T \tag{4.1}$$

where $y_{it} = r_{it} - r_{ft}$ are the excess returns in period t of asset i over the riskless asset r_{ft} (usually proxied by a short term interest rate) and $x_t = r_{mt} - r_{ft}$ are the excess returns in period t of the market portfolio (usually proxied by returns on an all-share index). The *CAPM* predicts that the intercepts α_i should be zero. Stacking the observations over t, $t = 1, \cdots, T$ we have the n equations

$$\mathbf{y}_i = \alpha_i + \beta_i \mathbf{x} + \mathbf{u}_i, \qquad i = 1, \cdots, n. \tag{4.2}$$

The $T \times 1$ disturbance vectors $\mathbf{u}_i$ are assumed to have the properties that

$$\mathrm{E}(\mathbf{u}_i) = \mathbf{0}$$

and

$$\mathrm{E}(\mathbf{u}_i \mathbf{u}_j') = \sigma_{ij} \mathbf{I}_t. \tag{4.3}$$

This is a particular case of a *SURE* model where the regressors are the same for all n equations and in which *OLS* is an efficient estimator. Thus the equations (4.2) can be estimated separately, without loss of information.

Consider *maximum likelihood* estimation of all the n equations jointly. Stacking the observations in (4.1) over assets i, $i = 1, \cdots, n$ we have, for the tth observation:

$$\mathbf{y}_t = \boldsymbol{\alpha} + \boldsymbol{\beta} x_t + \mathbf{u}_t \tag{4.4}$$

where $\mathbf{y}_t$ is an $n \times 1$ vector of observations on the excess returns of each asset and $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are $n \times 1$ vectors of parameters with

$$\mathrm{E}(\mathbf{u}_t) = \mathbf{0} \quad \text{and} \quad \mathrm{E}(\mathbf{u}_t \mathbf{u}_t') = \boldsymbol{\Sigma}.$$

Each time observation is, by assumption, independently normally distributed and so the likelihood function is given by

$$\begin{aligned} L(\mathbf{y}_t | x_t) &= \prod_{t=1}^{T} (2\pi)^{-\frac{n}{2}} |\boldsymbol{\Sigma}|^{-\frac{1}{2}} \exp(-\frac{1}{2}(\mathbf{y}_t - \boldsymbol{\alpha} - \boldsymbol{\beta} x_t)' \boldsymbol{\Sigma}^{-1} (\mathbf{y}_t - \boldsymbol{\alpha} - \boldsymbol{\beta} x_t)) \\ &= (2\pi)^{-\frac{nT}{2}} |\boldsymbol{\Sigma}|^{-\frac{T}{2}} \exp(-\frac{1}{2} \sum_{t=1}^{T} (\mathbf{y}_t - \boldsymbol{\alpha} - \boldsymbol{\beta} x_t)' \boldsymbol{\Sigma}^{-1} (\mathbf{y}_t - \boldsymbol{\alpha} - \boldsymbol{\beta} x_t)) \end{aligned}$$

and its logarithm is

$$\begin{aligned} \log L(\mathbf{y}; \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\Sigma} | \mathbf{x}) &= -\frac{nT}{2} \log(2\pi) - \frac{T}{2} \log |\boldsymbol{\Sigma}| \\ &\quad -\frac{1}{2} \sum_{t=1}^{T} (\mathbf{y}_t - \boldsymbol{\alpha} - \boldsymbol{\beta} x_t)' \boldsymbol{\Sigma}^{-1} (\mathbf{y}_t - \boldsymbol{\alpha} - \boldsymbol{\beta} x_t). \end{aligned} \tag{4.5}$$

The first order conditions for maximisation of (4.5) are

$$\frac{\partial \log L}{\partial \boldsymbol{\alpha}} = \boldsymbol{\Sigma}^{-1} \sum_{t=1}^{T} (\mathbf{y}_t - \boldsymbol{\alpha} - \boldsymbol{\beta} x_t) = \mathbf{0}$$

$$\frac{\partial \log L}{\partial \boldsymbol{\beta}} = \boldsymbol{\Sigma}^{-1} \sum_{t=1}^{T} [(\mathbf{y}_t - \boldsymbol{\alpha} - \boldsymbol{\beta} x_t) x_t] = \mathbf{0}$$

and

$$\frac{\partial \log L}{\partial \mathbf{\Sigma}^{-1}} = \frac{T}{2}\mathbf{\Sigma} - \frac{1}{2}\sum_{t=1}^{T}(\mathbf{y}_t - \boldsymbol{\alpha} - \boldsymbol{\beta}x_t)(\mathbf{y}_t - \boldsymbol{\alpha} - \boldsymbol{\beta}x_t)' = \mathbf{0}.$$

These equations can be solved to give the maximum likelihood estimators

$$\widehat{\boldsymbol{\alpha}} = \overline{\mathbf{y}} - \widehat{\boldsymbol{\beta}}\overline{x} \tag{4.6}$$

$$\widehat{\boldsymbol{\beta}} = \frac{\sum_{t=1}^{T}(x_t - \overline{x})(\mathbf{y}_t - \overline{\mathbf{y}})}{\sum_{t=1}^{T}(x_t - \overline{x})^2} \tag{4.7}$$

and

$$\widehat{\mathbf{\Sigma}} = \frac{1}{T}\sum_{t=1}^{T}(\mathbf{y}_t - \widehat{\boldsymbol{\alpha}} - \widehat{\boldsymbol{\beta}}x_t)(\mathbf{y}_t - \widehat{\boldsymbol{\alpha}} - \widehat{\boldsymbol{\beta}}x_t)' \tag{4.8}$$

where $\overline{\mathbf{y}} = \frac{1}{T}\sum_{t=1}^{T}\mathbf{y}_t$ and $\overline{x} = \frac{1}{T}\sum_{t=1}^{T}x_t$ are the sample means of $\mathbf{y}_t$ and x_t respectively.

It can be seen that the maximum likelihood estimators of $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ defined by (4.6) and (4.7) are just the *OLS* estimators and can be computed by separate *OLS* for each asset. The maximum likelihood estimator of $\mathbf{\Sigma}$ defined by (4.8) has typical element

$$\widehat{\sigma}_{ij} = \frac{\mathbf{e}_i'\mathbf{e}_j}{T}$$

where $\mathbf{e}_i = \mathbf{y}_i - \widehat{\alpha}_i - \widehat{\beta}_i\mathbf{x}$ is the $T \times 1$ vector of *OLS* residuals from estimation of the equation for asset i.

4.2 Estimating the Black zero-beta *CAPM*

In the Black version of the *CAPM*, there is no risk-free asset and the expected return on the zero-beta portfolio, $\mathrm{E}(r_0)$ in (3.1), is not directly observable. Instead it is treated as an unknown model parameter. From (3.1) we have

$$\mathrm{E}(r_i) = (1 - \beta_i)\,\mathrm{E}(r_0) + \beta_i\,\mathrm{E}(r_m)$$

which gives rise to the regression equation

$$y_{it} = \alpha_i + \beta_i x_t + u_{it}, \qquad i = 1, \cdots, n, \quad t = 1, \cdots, T \tag{4.9}$$

where now $y_{it} = r_{it}$ is the return in period t of asset i, $x_t = r_{mt}$ is the return in period t of the market portfolio and the intercept α_i is defined by

$$\alpha_i = (1 - \beta_i)\gamma \tag{4.10}$$

where γ estimates the expected return of the zero-beta portfolio, $\mathrm{E}(r_0)$. Note that the model implies that α_i is proportional to $1-\beta_i$, with the same constant of proportionality γ for all assets i. This is a nonlinear restriction and to take this into account, we need to estimate all the equations jointly.

If the nonlinear parameter restriction in (4.10) is ignored, then estimation of α_i and β_i proceeds as before, except that the variables are now real returns rather than excess returns. In particular, *OLS* will still be an efficient estimator. When the nonlinear restriction is taken into account, then the log-likelihood function of the system is given by

$$\begin{aligned}\log L(\mathbf{y};\gamma,\boldsymbol{\beta},\boldsymbol{\Sigma}|\mathbf{x}) &= -\frac{nT}{2}\log(2\pi)-\frac{T}{2}\log|\boldsymbol{\Sigma}| \\ &\quad -\frac{1}{2}\sum_{t=1}^{T}(\mathbf{y}_t-\gamma(\boldsymbol{\iota}-\boldsymbol{\beta})-\boldsymbol{\beta}x_t)'\boldsymbol{\Sigma}^{-1}(\mathbf{y}_t-\gamma(\boldsymbol{\iota}-\boldsymbol{\beta})-\boldsymbol{\beta}x_t)\end{aligned} \tag{4.11}$$

with first order conditions

$$\frac{\partial \log L}{\partial \gamma} = (\boldsymbol{\iota}-\boldsymbol{\beta})'\boldsymbol{\Sigma}^{-1}\sum_{t=1}^{T}(\mathbf{y}_t-\gamma(\boldsymbol{\iota}-\boldsymbol{\beta})-\boldsymbol{\beta}x_t)=0$$

$$\frac{\partial \log L}{\partial \boldsymbol{\beta}} = \boldsymbol{\Sigma}^{-1}\sum_{t=1}^{T}[(\mathbf{y}_t-\gamma(\boldsymbol{\iota}-\boldsymbol{\beta})-\boldsymbol{\beta}x_t)(x_t-\gamma)]=\mathbf{0}$$

and

$$\frac{\partial \log L}{\partial \boldsymbol{\Sigma}^{-1}} = \frac{T}{2}\boldsymbol{\Sigma}-\frac{1}{2}\sum_{t=1}^{T}(\mathbf{y}_t-\gamma(\boldsymbol{\iota}-\boldsymbol{\beta})-\beta x_t)(\mathbf{y}_t-\gamma(\boldsymbol{\iota}-\boldsymbol{\beta})-\boldsymbol{\beta}x_t)'=\mathbf{0}.$$

Solving the first order conditions gives the expressions

$$\widetilde{\gamma} = \frac{(\boldsymbol{\iota}-\widetilde{\boldsymbol{\beta}})'\widetilde{\boldsymbol{\Sigma}}^{-1}(\overline{\mathbf{y}}-\widetilde{\boldsymbol{\beta}}\overline{x})}{(\boldsymbol{\iota}-\widetilde{\boldsymbol{\beta}})'\widetilde{\boldsymbol{\Sigma}}^{-1}(\boldsymbol{\iota}-\widetilde{\boldsymbol{\beta}})} \tag{4.12}$$

$$\widetilde{\boldsymbol{\beta}} = \frac{\sum_{t=1}^{T}(x_t-\widetilde{\gamma})(\mathbf{y}_t-\widetilde{\gamma}\boldsymbol{\iota})}{\sum_{t=1}^{T}(x_t-\widetilde{\gamma})^2} \tag{4.13}$$

and

$$\widetilde{\boldsymbol{\Sigma}} = \frac{1}{T}\sum_{t=1}^{T}(\mathbf{y}_t-\widetilde{\gamma}(\boldsymbol{\iota}-\widetilde{\boldsymbol{\beta}})-\widetilde{\boldsymbol{\beta}}x_t)(\mathbf{y}_t-\widetilde{\gamma}(\boldsymbol{\iota}-\widetilde{\boldsymbol{\beta}})-\widetilde{\boldsymbol{\beta}}x_t)'. \tag{4.14}$$

The three equations (4.12), (4.13) and (4.14) must be solved jointly to derive the maximum likelihood estimators. One solution algorithm would be to iterate around the equations until convergence. Alternatively, the *ML* estimators can be found directly by a nonlinear search for the maximum of (4.11).

References

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