Financial Econometrics
Lecture 6: Testing the CAPM model

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1 Introduction

The capital asset pricing model has some strong implications which are testable. The restrictions that can be tested depend on the version of the *CAPM* that has been estimated, the standard *Sharpe-Lintner* version of Sharpe (1964), Lintner (1965) and Mossin (1966) or the *zero-beta* version of Black (1972).

2 Testing the Sharpe-Lintner *CAPM*

Recall from last week that the standard form of the *CAPM* results in the estimation equations

$$\mathbf{y}_i = \alpha_i + \beta_i \mathbf{x} + \mathbf{u}_i \qquad i = 1, \cdots, n \tag{2.1}$$

where $\mathbf{y}_i = \mathbf{r}_i - \mathbf{r}_f$ is the $T \times 1$ vector of excess returns of asset i over the riskless asset $\mathbf{r}_f$ (usually proxied by a short term interest rate) and $\mathbf{x} = \mathbf{r}_m - \mathbf{r}_f$ is the $T \times 1$ vector of excess returns of the market portfolio (usually proxied by returns on an all-share index).

The key testable restriction in the Sharpe-Lintner version of the *CAPM* is that the intercepts α_i are all zero. This is essentially a test that the market portfolio is an efficient portfolio, which is a key assumption of the model. This hypothesis can be tested for a single asset i by a standard t-test of the null hypothesis that $\alpha_i = 0$ against the alternative $\alpha_i \neq 0$ in the single equation (2.1). More interesting is the joint test on the $n \times 1$ vector α of the null hypothesis that $\alpha = \mathbf{0}$. To compute this test, we must have estimated the complete *SURE* system

$$\mathbf{y}_t = \boldsymbol{\alpha} + \boldsymbol{\beta} x_t + \mathbf{u}_t \tag{2.2}$$

by maximum likelihood, where $\mathbf{y}_t$ is an $n \times 1$ vector of observations on the excess returns of each asset and α and $\boldsymbol{\beta}$ are $n \times 1$ vectors of parameters with

$$\mathrm{E}(\mathbf{u}_t) = \mathbf{0} \quad \text{and} \quad \mathrm{E}(\mathbf{u}_t \mathbf{u}_t') = \boldsymbol{\Sigma}.$$

Then an asymptotically valid *Wald* test is given by the statistic

$$W = \widehat{\alpha}'(\text{var}(\widehat{\alpha}))^{-1}\widehat{\alpha} = T\frac{\widehat{\alpha}'\widehat{\mathbf{\Sigma}}^{-1}\widehat{\alpha}}{(1+\frac{\overline{\mathbf{x}}^2}{\widehat{\sigma}_m^2})} \sim_a \chi_n^2 \tag{2.3}$$

where $\widehat{\boldsymbol{\alpha}}$, $\widehat{\mathbf{\Sigma}}$ and $\overline{\mathbf{x}}$ are defined by equations

$$\widehat{\boldsymbol{\alpha}} = \overline{\mathbf{y}} - \widehat{\boldsymbol{\beta}}\overline{x} \tag{2.4}$$

$$\widehat{\boldsymbol{\beta}} = \frac{\sum_{t=1}^{T}(x_t - \overline{x})(\mathbf{y}_t - \overline{\mathbf{y}})}{\sum_{t=1}^{T}(x_t - \overline{x})^2} \tag{2.5}$$

and

$$\widehat{\mathbf{\Sigma}} = \frac{1}{T}\sum_{t=1}^{T}(\mathbf{y}_t - \widehat{\boldsymbol{\alpha}} - \widehat{\boldsymbol{\beta}}x_t)(\mathbf{y}_t - \widehat{\boldsymbol{\alpha}} - \widehat{\boldsymbol{\beta}}x_t)' \tag{2.6}$$

from last week and $\widehat{\sigma}_m^2$ is defined by the equation

$$\widehat{\sigma}_m^2 = \frac{1}{T}\sum_{t=1}^{T}(x_t - \overline{x})^2.$$

On the null hypothesis that $\alpha = \mathbf{0}$, the statistic (2.3) is distributed *asymptotically* as a chi-squared with n degrees of freedom. MacKinlay (1987) shows that an alternative, exact form of this statistic is available and is defined by

$$\frac{T-n-1}{nT}W = \frac{T-n-1}{n}\frac{\widehat{\alpha}'\widehat{\mathbf{\Sigma}}^{-1}\widehat{\alpha}}{(1+\frac{\overline{\mathbf{x}}^2}{\widehat{\sigma}_m^2})} \sim F_{n,T-n-1.}$$

Under the null, this statistic is *exactly* distributed with an F distribution with degrees of freedom, n and $T-n-1$.

3 Testing the Black zero-beta *CAPM*

The key testable restriction in the *Black zero-beta* form of the *CAPM* is the non-linear restiction that

$$H_0 : \boldsymbol{\alpha} = (\boldsymbol{\iota} - \boldsymbol{\beta})\gamma.$$

This can be tested against the alternative hypothesis that

$$H_1 : \boldsymbol{\alpha} \neq (\boldsymbol{\iota} - \boldsymbol{\beta})\gamma.$$

On the alternative hypothesis, the unrestricted model can be efficiently estimated by *OLS* on

$$y_{it} = \alpha_i + \beta_i x_t + u_{it}, \qquad i = 1, \cdots, n, \quad t = 1, \cdots, T \tag{3.1}$$

On the null hypothesis, estimation involves maximising the log-likelihood function of the restricted model

$$\begin{aligned} \log L(\mathbf{y}; \gamma, \boldsymbol{\beta}, \boldsymbol{\Sigma}|\mathbf{x}) &= -\frac{nT}{2}\log(2\pi) - \frac{T}{2}\log|\boldsymbol{\Sigma}| \\ &-\frac{1}{2}\sum_{t=1}^{T}(\mathbf{y}_t - \gamma(\boldsymbol{\iota} - \boldsymbol{\beta}) - \boldsymbol{\beta}x_t)'\boldsymbol{\Sigma}^{-1}(\mathbf{y}_t - \gamma(\boldsymbol{\iota} - \boldsymbol{\beta}) - \boldsymbol{\beta}x_t). \end{aligned} \tag{3.2}$$

If both the unrestricted and restricted versions of the model are estimated, then the null hypothesis can be tested by the likelihood ratio test

$$\lambda = -2\log(\frac{L_r}{L_u}) = 2(\log L_u - \log L_r) \sim_a \chi^2_{n-1}$$

where L_u is the maximum of the likelihood of the unrestricted model and L_r is the maximum of the likelihood of the restricted model. Note that the number of restrictions (and hence the degrees of freedom parameter) is $n-1$ because the scalar parameter γ is estimated on the null hypothesis. This test was developed by Gibbons (1982) and Shanken (1985). It can also be written as

$$T(\log\left|\widehat{\boldsymbol{\Sigma}}\right| - \log\left|\widetilde{\boldsymbol{\Sigma}}\right|) \sim_a \chi^2_{n-1}$$

where $\widehat{\boldsymbol{\Sigma}}$ and $\widetilde{\boldsymbol{\Sigma}}$ are defined by (2.6) and

$$\widetilde{\boldsymbol{\Sigma}} = \frac{1}{T}\sum_{t=1}^{T}(\mathbf{y}_t - \widetilde{\gamma}(\boldsymbol{\iota} - \widetilde{\boldsymbol{\beta}}) - \widetilde{\boldsymbol{\beta}}x_t)(\mathbf{y}_t - \widetilde{\gamma}(\boldsymbol{\iota} - \widetilde{\boldsymbol{\beta}}) - \widetilde{\boldsymbol{\beta}}x_t)'. \tag{3.3}$$

respectively.

4 Cross-sectional tests of the *CAPM*

The standard form of the *CAPM* predicts that there is a linear relationship

$$r_{it} = r_{ft} + (r_{mt} - r_{ft})\beta_i + u_{it} \tag{4.1}$$

between excess returns $r_i - r_f$ for asset i and the parameter β_i, defined by

$$\beta_i = \frac{\text{cov}(r_i, r_m)}{\text{var}(r_m)},$$

which is a measure of the relative riskiness of that asset in the portfolio. This linear relationship is assumed to hold at each point in time. To estimate the β_i parameters requires time-series data but, because of the special *SURE* structure of the model, each β_i can be estimated independently. A natural question is whether these estimated β_i parameters are positively and linearly related to excess returns over the cross-section.

Averaging equation (4.1) over time we have

$$\overline{r}_i = \overline{r}_f + (\overline{r}_m - \overline{r}_f)\beta_i + \overline{u}_i$$

where $\overline{z}_i = \frac{1}{T}\sum_{t=1}^{T} z_{it}$. This suggests the regression equation

$$\overline{\mathbf{r}} = a_0 + a_1\widehat{\boldsymbol{\beta}} + \mathbf{v} \tag{4.2}$$

where $\overline{\mathbf{r}}$ is the $n \times 1$ vector of observations on $\overline{r}_i$, $\widehat{\boldsymbol{\beta}}$ is the $n \times 1$ vector of estimates of $\beta_i i$ obtained from time-series regression, a_0 and a_1 are parameters and $\mathbf{v}$ is a disturbance vector. Clearly, if the *CAPM* holds, then $a_0 = \overline{r}_f$ and $a_1 = \overline{r}_m - \overline{r}_f$. This is a *cross-sectional regression* using, as a regressor, the vector of parameter estimates *bbetahat*. This is a case of a *generated regressor* and means that there will be measurement error. Furthermore, because of the assumption

$$\mathrm{E}(\mathbf{u}_i\mathbf{u}_j') = \sigma_{ij}\mathbf{I}_t. \tag{4.3}$$

on the original disturbances, the disturbances $\mathbf{v}$ will be both heteroscedastic and non-diagonal. These problems can be alleviated to some extent by grouping the assets into portfolios and testing the *CAPM* on the portfolios.

Fama and MacBeth (1974) ran a form of this cross-sectional regression using data on twenty portfolios, adding additional regressors to test for nonlinearity. They found some support for the restrictions implied by the *CAPM*.

References

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