Financial Econometrics
Lecture 8: Nonlinear models in Financial Econometrics

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1 Introduction

Nonlinear models have become increasingly popular in finance. This is partly a recognition of the inherent nonlinearity of much financial data and partly a response to the increasing tractability of nonlinear models with recent explosions in computing power. The class of nonlinear models is very large but one family of models has received special attention: *switching regime* models. In this lecture we look at two of the simplest models in this family, the *SETAR* and the *STAR* model while in next week's lecture we look at the *Markov switching model* of Hamilton as well as the semi-parametric artifical neural network *ANN* model which bears some resemblances to the *STAR* model. Franses and van Dijk (2000) gives an excellent overview of nonlinear models in empirical finance while Granger and Teräsvirta (1993) takes a more general look at the use of nonlinear models in economics.

2 Equilibrium in nonlinear models

Consider the general *nonlinear* model

$$y_t = f_t(y_{t-1}, u_t, \boldsymbol{\theta})$$

where $f_t()$ is a *nonlinear* function of y_{t-1} and a random disturbance u_t with parameters represented by the vector $\boldsymbol{\theta}$. The deterministic part of the model, $f_t(y_{t-1}, 0, \boldsymbol{\theta})$ is found by setting the disturbances to zero and is called the *skeleton* of the model. The model has an *equilbrium* at the point y_t^* if y_t^* is a *fixed point* of the *skeleton* so that

$$y_t^* = f(y_t^*, 0, \boldsymbol{\theta}).$$

This *equibrium* is said to be *stable* if y_t converges to y_t^* in the absence of shocks u_t. Otherwise it is *unstable*. A *stable equilibrium* can either be *globally stable* if

y_t converges to y_t^* from any starting point or *locally stable*, if y_t only converges to y_t^* from certain starting points. A *stable equilibrium* is also known as an *attractor* because y_t is attracted to y_t^*, and, when that equilibrium is *locally stable*, then the region in which the process converges to an *attractor* is known as the *domain of attraction.*

In the linear model

$$y_t = \theta_0 + \theta_1 y_{t-1} + u_t$$

there is a *unique equilibrium* at

$$y_t^* = \frac{\theta_0}{1-\theta_1}$$

as long as $\theta_1 \neq 1$. This equilibrium is *globally stable* if $|\theta_1| < 1$ but otherwise it is *unstable.* When $\theta_1 = 1$ (a positive unit root), the skeleton is

$$y_t = \theta_0 + y_{t-1}$$

and an *equilibrium* exists only if $\theta_0 = 0$, otherwise there is no equilibrium. When $\theta_0 = 0$, there is an infinite number of equilbria since any value of y_t is an equilibrium.

With *nonlinear* models, there can often be *multiple equilibria*, some of which will be *stable* and some *unstable.* For example, the simple cubic function

$$y_t = y_{t-1}^3 + u_t$$

has three equilibria at $y_t^* = -1$, $y_t^* = 0$, and $y_t^* = 1$, but only the middle one is stable.

3 Regime Switching Models

One important class of nonlinear models that have proved popular in the finance literature are the *threshold autoregressive* (*TAR*) models of Tong (1978) and Tong and Lim (1980). In these models there are two or more regimes and a mechanism whereby the model switches between them according to whether a particular variable reaches a threshold value. The most important models in this class are the *SETAR* model and the *STAR* model. Another important model is the Markov Switching Regimes model, in which the regime is unobserved but assumed to follow a simple statistical process.

3.1 The *SETAR* model

The *Self-Exciting Threshold Autoregressive* or *SETAR* model is defined by

$$y_t = \begin{cases} \alpha_{01} + \alpha_{11} y_{t-1} + u_t, & y_{t-d} \leq c \\ \alpha_{02} + \alpha_{12} y_{t-1} + u_t, & y_{t-d} > c \end{cases}. \tag{3.1}$$

where the disturbance u_t is assumed to be independently, identically distributed $u_t \sim iid(0, \sigma^2)$. In this model, the process generating y_t switches between two regimes, depending on whether y_{t-d} is greater or less than the threshold value c, where d represents the *delay* in response. When $|\alpha_{11}| < 1$ and $|\alpha_{11}| < 1$, then the model in each regime is a linear autoregressive process with mean given by

$$\mu_1 = \frac{\alpha_{01}}{1 - \alpha_{11}}$$

in regime 1, when $y_{t-d} \leq c$, and

$$\mu_2 = \frac{\alpha_{02}}{1 - \alpha_{12}}$$

in regime 2, when $y_{t-d} > c$. The nonlinearity of the model is caused by the endogenous switching betwen the two regimes.

The *SETAR* model (3.1) can be written in the alternative form

$$y_t = (\alpha_{01} + \alpha_{11} y_{t-1})(1 - I(y_{t-d} > c)) + (\alpha_{02} + \alpha_{12} y_{t-1}) I(y_{t-d} > c) + u_t \tag{3.2}$$

where $I(a)$ is an indicator function with $I(a) = 1$ when the condition a is true and $I(a) = 0$ when a is false. The *skeleton* of (3.2) is

$$y_t = (\alpha_{01} + \alpha_{11} y_{t-1})(1 - I(y_{t-d} > c)) + (\alpha_{02} + \alpha_{12} y_{t-1}) I(y_{t-d} > c)$$

which is just the conditional expectation $\mathrm{E}(y_t | y_{t-1}, y_{t-d})$.

The way that the *SETAR* model behaves depends on the values of the parameters. To give an illustration, suppose that $c = 0$ and $d = 1$ so that the model is in the second regime whenever y_{t-1} is positive and in the first regime otherwise. Setting the autoregressive coefficients to $\alpha_{11} = -0.5$ and $\alpha_{12} = 0.5$, we consider different values of the intercept terms α_{01} and α_{02}. Four cases can be distinguished:

α_{01}	α_{02}	*equilibria*	*stability*
0	0	$y_t^* = 0$	*stable*
-0.3	-0.2	$y_t^* = -0.2$	*stable*
-0.3	0.1	$y_t^* = -0.2$; $y_t^* = 0.2$	*both stable*
0.3	-0.1	*none*	*limit cycle*

In the first case, when both intercepts are zero, the two regimes have the same mean and there is a single *globally stable equilibrium* at zero. In the second case, there is also a single stable equilibrium. This is because the mean in both regimes is negative and so, whenever the model is in the second regime, it is attracted back to the first regime. In the third case, there are two *locally stable equilibria* at -0.2 and at 0.2. This is because the mean in the first regime is negative while the mean in the second regime is positive. In the absence of shocks, there is nothing ever to cause the process to switch regimes so that it will converge to the mean of whichever regime it starts from. Finally, in the fourth case, there is *no equilibrium*. This is because the mean in the first regime is positive while the mean in the second regime is negative, so that whenever the model is in one regime, it is attracted to the other regime. The model will cycle indefinitely between the three values $y_1^* = \frac{1}{15}$, $y_2^* = -\frac{1}{15}$ and $y_3^* = \frac{1}{3}$. This set of points is known as a *limit cycle* and can be regarded as the attractor of the process. Even in the absence of *exogenous stochastic shocks*, this model displays *endogenous dynamics*.

Some deterministic nonlinear processes can have *chaotic dynamics* where the slightest change in the starting values can change the time path of the process. An example is the well-known *tent map*, which is the deterministic *SETAR* model

$$y_t = \begin{cases} 0 + 2y_{t-1}, & y_{t-1} < 0.5 \\ 2 - 2y_{t-1}, & y_{t-1} \geq 0.5 \end{cases}. \tag{3.3}$$

for $y_0 \in (0, 1)$. Despite the fact that there is no disturbance in this model, it generates sequences of observations that appear to be random and are uniformly distributed on the unit interval and serially uncorrelated.

The *SETAR* model can be generalised to higher order *AR* processes but little is known in general about the conditions under which such models are *stationary*. For the first order case of equation (3.1) where $d = 1$, Chan *et al.* (1985) prove that the model is stationary *if and only if* one of the following five conditions is satisfied:

1	$\alpha_{11} < 1, \alpha_{12} < 1, \alpha_{11}\alpha_{12} < 1$
2	$\alpha_{11} = 1, \alpha_{12} < 1, \alpha_{01} > 0$
3	$\alpha_{11} < 1, \alpha_{12} = 1, \alpha_{02} < 0$
4	$\alpha_{11} = 1, \alpha_{12} = 1, \alpha_{02} < 0 < \alpha_{01}$
5	$\alpha_{11}\alpha_{12} = 1, \alpha_{11} < 0, \alpha_{02} + \alpha_{12}\alpha_{01} > 0.$

Note that conditions 2-4 correspond to cases when one or both of the regimes has a unit root and so is non-stationary, but the model is still globally stationary since the intercept conditions mean that the model is attracted to a stationary regime.

3.2 The *STAR* model

In the *SETAR* model (3.2), there is an abrupt switch from one regime to the other at the threshold value $y_{t-d} = c$. An alternative model to (3.2) is

$$y_t = (\alpha_{01} + \alpha_{11} y_{t-1})(1 - S(y_{t-d})) + (\alpha_{02} + \alpha_{12} y_{t-1}) S(y_{t-d}) + u_t \qquad (3.4)$$

where $S()$ is a *transition function* varying smoothly between between 0 and 1 as y_{t-d} increases. This is the *Smooth Transition Autoregressive* or *STAR* model of Chan and Tong (1986), Granger and Teräsvirta (1993) and Teräsvirta (1994). A popular choice for $S()$ is the *logistic function*

$$S(y_{t-d}, \gamma, c) = \frac{1}{1 + \exp(-\gamma(y_{t-d} - c))} \qquad (3.5)$$

in which case the model is called a *Logistic STAR* or *LSTAR* model. The parameter γ in (3.5) determines the smoothness of the transition from one regime to another.As $\gamma \to \infty$, the logistic function tends to a step function and the *LSTAR* model tends to the *SETAR* model. As $\gamma \to 0$, the transition function $S()$ tends to the constant value, 0.5, and the model becomes the linear autoregressive model

$$y_t = \frac{\alpha_{01} + \alpha_{02}}{2} + \frac{\alpha_{11} + \alpha_{12}}{2} y_{t-1} + u_t.$$

With positive values of γ, $\gamma \in (0, \infty)$, the model is a linear combination of the two regimes

$$y_t = ((1 - w_t)\alpha_{01} + w_t \alpha_{02}) + ((1 - w_t)\alpha_{11} + w_t \alpha_{12}) y_{t-1} + u_t$$

where the weights w_t will be time-varying.

4 Regime switching models of volatility

The nonlinear models considered so far allow for nonlinearity in the mean. It is also possible to consider nonlinear versions of models in the *GARCH* family. In particular, regime switching models of conditional variance allow volatility to have an asymmetric response to positive and negative shocks. Fornari and Mele (1996, 1997) proposed the *Volatility Switching GARCH* or *VS-GARCH* model in which conditional variance σ_t^2 follows a *SETAR* model

$$\sigma_t^2 = \begin{cases} \alpha_{01} + \alpha_{11} u_{t-1}^2 + \beta_{11} \sigma_{t-1}^2, & u_{t-1} \leq 0 \\ \alpha_{02} + \alpha_{12} u_{t-1}^2 + \beta_{12} \sigma_{t-1}^2, & u_{t-1} > 0 \end{cases}. \qquad (4.1)$$

or

$$\begin{aligned}\sigma_t^2 &= (\alpha_{01} + \alpha_{11}u_{t-1}^2 + \beta_{11}\sigma_{t-1}^2)(1 - I(u_{t-1})) \\ &\quad +(\alpha_{02} + \alpha_{12}u_{t-1}^2 + \beta_{12}\sigma_{t-1}^2)I(u_{t-1}). \end{aligned} \tag{4.2}$$

The unconditional variance of u_t in this model is given by

$$\sigma^2 = \frac{(\alpha_{01} + \alpha_{02})/2}{1 - (\alpha_{11} + \alpha_{12})/2 - (\beta_{11} + \beta_{12})/2}.$$

The kurtosis in this model is higher than in the standard *GARCH(1,1)* model. With parameter restrictions $\alpha_{11} > \alpha_{12}$ and $\alpha_{01}+\beta_{11}\sigma_t^2 < \alpha_{02}+\beta_{12}\sigma_t^2$, small positive shocks have a larger impact on conditional volatility than small negative shocks but the reverse holds for large shocks. Thus the model allows more complicated asymmetric effects than other nonlinear models like the *EGARCH* model.

Anderson, Nam and Vahid (1999) consider an *Asymmetric Nonlinear Smooth Transition GARCH* or *ANST-GARCH* model

$$\begin{aligned}\sigma_t^2 &= (\alpha_{01} + \alpha_{11}u_{t-1}^2 + \beta_{11}\sigma_{t-1}^2)(1 - S(u_{t-1}, \gamma, 0)) \\ &\quad +(\alpha_{02} + \alpha_{12}u_{t-1}^2 + \beta_{12}\sigma_{t-1}^2)S(u_{t-1}, \gamma, 0) \end{aligned} \tag{4.3}$$

where $S(u_{t-1}, \gamma, 0)$ is the logistic function defined by equation (3.5).

5 Testing for nonlinearity

5.1 *RESET* tests

A test for linearity against a general nonlinear alternative is provided by the *RESET test* of Ramsey (1969). This is based on the regression

$$\widehat{u}_t = \mathbf{x}_t'\boldsymbol{\beta} + \sum_{j=2}^{q} \delta_j \widehat{y}_t^j + v_t \tag{5.1}$$

where $\widehat{y}_t$ are the fitted values and $\widehat{u}_t$ the residuals from the linear model

$$y_t = \mathbf{x}_t'\boldsymbol{\beta} + u_t. \tag{5.2}$$

The regression (5.1) includes nonlinear powers of $\widehat{y}_t$ which should have zero coefficients if the relationship between y_t and $\mathbf{x}_t$ is linear. Thus the null hypothesis is that $\delta_j = 0$, $j = 2, \cdots, q$ which can be tested by the F statistic

$$F = \frac{(RSS_1 - RSS_2)/r}{RSS_2/(T - k - r)} \sim F_{r,T-k-q+1} \tag{5.3}$$

where RSS_1 is the *Residual Sum of Squares*, $\sum_t \widehat{u}_t^2$, from (5.2), RSS_2 is the *Residual Sum of Squares*, $\sum_t \widehat{v}_t^2$, from (5.1) and $r = q - 1$ is the number of restrictions. The test is exact if $\mathbf{x}_t$ does not contain any lags of y_t and is asympototically valid otherwise. An equivalent asymptotic form is given by the chi-squared statistic

$$(T - r)R^2 \sim_a \chi^2(r)$$

where R^2 is the uncorrected goodness of fit measure from (5.1). This form of the statistic, for the case $r = 1$, is the misspecification diagnostic test reported by MicroFit.

A *modified RESET* test, due to Thursby and Schmidt (1977) is based on the alternative auxiliary regression

$$\widehat{u}_t = \mathbf{x}_t'\boldsymbol{\beta} + \sum_{j=2}^{q} \boldsymbol{\delta}_j'\overline{\mathbf{x}}_t^j + v_t \tag{5.4}$$

where $\overline{\mathbf{x}}_t^j$ is the $(k-1)\times 1$ vector of jth powers of each of the elements of $\mathbf{x}_t$, except the intercept. The null hypothesis is that $\boldsymbol{\delta}_j = 0$, for $j = 2, \cdots, q$. This implies $(k-1)(q-1)$ coefficient restrictions and leads to the test statistic (5.3) where now RSS_2 is the *Residual Sum of Squares*, $\sum_t \widehat{v}_t^2$, from (5.4) and $r = (k-1)(q-1)$. This *modified RESET* test allows the nonlinear effects to differ for different elements of $\mathbf{x}_t$, and would be expected to be more powerful in some circumstances.

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