Macroeconomics
Lecture 1: The Solow Growth Model

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1 Introduction

One of the most important long-run issues in macroeconomics is understanding growth. Why do economies grow and what determines how fast they grow? In this lecture we look at a key model of growth developed independently by Robert Solow (1956) and Trevor Swan (1956) and known variously as the Solow growth model, the Solow-Swan growth model or the neo-classical growth model.

2 The Production Function

The heart of the theory of growth is a production function relating output to the inputs necessary to create it, namely the factors of production. The production function is written as

$$Y(t) = F(K(t), L(t)) \tag{2.1}$$

where Y is output, K is the input of physical capital (buildings and machines) and L is the input of labour. It will be assumed for simplicity that each worker supplies a single unit of labour so that L also represents the number of workers. The function $F()$ represents the production process that transforms the factors of production into output. Note that since we are considering output of the whole economy, this is an *aggregate* production function. The simplest assumption that validates the existence of such a production function for the whole economy is that the economy comprises a single good. Alternatively, if we allow for many goods in the economy, we must assume that we can aggregate the production functions for each good into a valid production function for the whole economy. All three variables in (2.1) are functions of time, t, and it is the way in which the factors of production grow over time which will determine the growth of output. Note that Y is a flow variable representing the rate of flow of output while K and L are both stocks of the factors of production, capital and labour, existing at a point in time. For convenience, the time argument (t) will henceforth be dropped and the production function rewritten as

$$Y = F(K, L) \tag{2.2}$$

We make the following assumptions on the derivatives of $F()$:

$$\partial F/\partial K = F_K > 0 \quad \partial F/\partial L = F_L > 0 \tag{2.3}$$

$$\partial^2 F/\partial K^2 = F_{KK} < 0 \quad \partial^2 F/\partial L^2 = F_{LL} < 0 \tag{2.4}$$

and

$$\partial^2 F/\partial K \partial L = F_{KL} > 0.$$

The first two conditions simply state that output is increasing in each of the factor inputs whereas the second two conditions embody the standard assumption of diminishing marginal product. The final condition states that when the input of one factor increases, the marginal product of the other factor increases.

Another important assumption that we make is that the production function exhibits *constant returns to scale*. What this means is that, for any $\lambda > 0$,

$$\lambda Y = F(\lambda K, \lambda L) \tag{2.5}$$

Constant returns to scale implies that if both the inputs double, then output doubles. It can be justified by the argument of replication, that when capital and labour inputs are doubled, they are used in the same way as before and so output doubles. This implies that the economy is large enough so that the gains from specialisation have been exhausted. The constant returns to scale assumption allows us to work with the production function in intensive form. Setting $\lambda = 1/L$ we have

$$Y/L = F(K/L, 1) \tag{2.6}$$

or

$$y = f(k) \tag{2.7}$$

where $y = Y/L$ is output per worker (per capita) and $k = K/L$ is capital per worker (the capital-labour ratio).

Since

$$f(k) = F(K, L)/L \tag{2.8}$$

and

$$\frac{\partial K}{\partial k} = \frac{\partial (kL)}{\partial k} = L \tag{2.9}$$

it follows from (2.3) and (2.4) and the chain rule that

$$\frac{\partial f}{\partial k} = f'(k) = F_K \frac{\partial K}{\partial k}/L = F_K > 0 \tag{2.10}$$

and

$$\frac{\partial^2 f}{\partial k^2} = f''(k) = \frac{\partial f'(k)}{\partial k} = F_{KK}\frac{\partial K}{\partial k} = F_{KK}L < 0. \tag{2.11}$$

The intensive-form production function is also assumed to satisfy the Inada conditions (Inada (1964)) that:

$$\lim_{k\to 0} f'(k) = \infty \quad \text{and} \quad \lim_{k\to\infty} f'(k) = 0. \tag{2.12}$$

Intuitively, these conditions state that when k is sufficiently small, its marginal product is very large but that as k becomes large, its marginal product approaches zero. The rôle of these conditions is to ensure that the path of the economy does not diverge. A production function satisfying the conditions (2.10), (2.11) and (2.12) is shown in Figure 1.

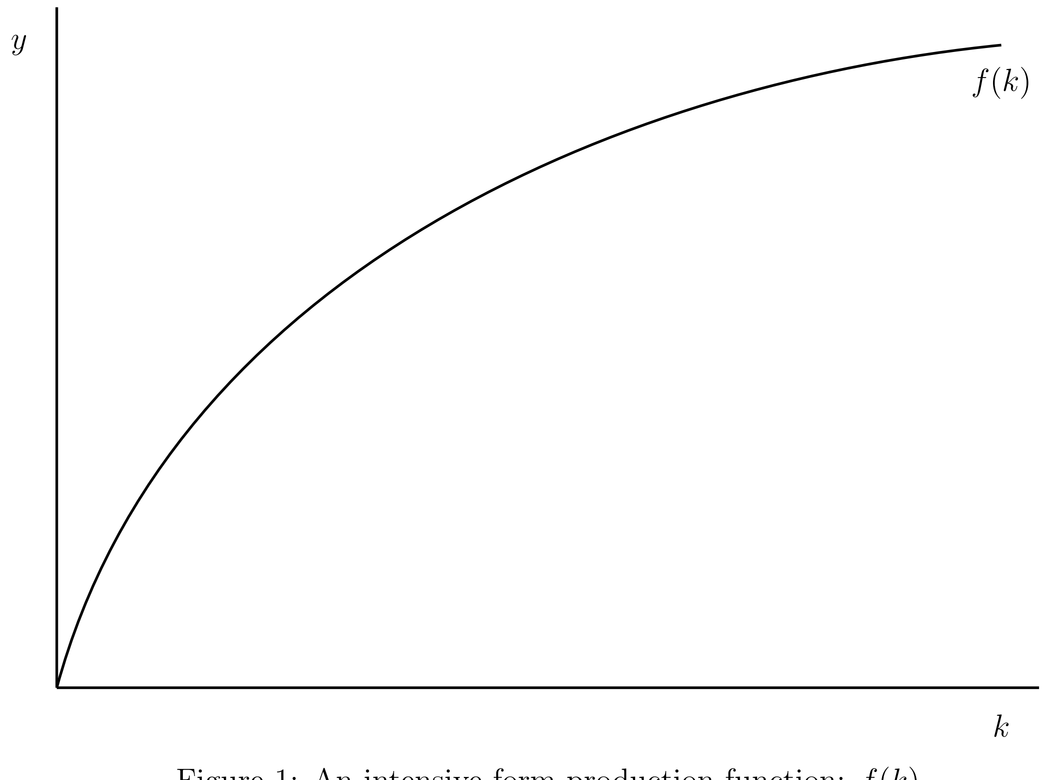

Figure 1: An intensive-form production function: $f(k)$

2.1 The Cobb-Douglas Production Function

A specific form of production function commonly used in the literature is the Cobb-Douglas production function (Cobb and Douglas (1928)). This takes the

form:
$$F(K,L) = K^\alpha L^\beta \tag{2.13}$$
with $\alpha > 0$ and $\beta > 0$. For this production function
$$F(\lambda K, \lambda L) = (\lambda K)^\alpha (\lambda L)^\beta = \lambda^{\alpha+\beta} K^\alpha L^\beta = \lambda^{\alpha+\beta} F(K,L) \tag{2.14}$$
so that, for constant returns to scale, we require that $\alpha + \beta = 1$ or $\beta = 1 - \alpha$. Imposing this condition, the constant returns to scale form of the Cobb-Douglas production function can be written as
$$F(K,L) = K^\alpha L^{1-\alpha} \tag{2.15}$$
with $0 < \alpha < 1$ or, in intensive form, as
$$f(k) = F(K/L, 1) = (K/L)^\alpha = k^\alpha. \tag{2.16}$$
Note that this production function satisfies the conditions (2.10) and (2.11) since
$$f'(k) = \alpha k^{\alpha-1} > 0 \tag{2.17}$$
and
$$f''(k) = \alpha(\alpha-1)k^{\alpha-2} = -\alpha(1-\alpha)k^{\alpha-2} < 0. \tag{2.18}$$
It also can be seen that it satisfies the Inada conditions (2.12).

The Cobb-Douglas function is an example of a production function with a *constant elasticity of substitution* between the factors of production K and L. The elasticity of substitution is a measure of the curvature of the isoquants. For the Cobb-Douglas production function, the slope of the isoquants is
$$Q = \frac{\partial F/\partial K}{\partial F/\partial L} = \frac{\alpha K^{\alpha-1}L^{1-\alpha}}{(1-\alpha)K^\alpha L^{-\alpha}} = \frac{\alpha}{(1-\alpha)} L/K \tag{2.19}$$
and the elasticity is given by
$$\left[\frac{\partial Q}{\partial (L/K)} . \frac{L/K}{Q}\right]^{-1} = \left[\frac{\alpha}{(1-\alpha)} \frac{L/K}{\frac{\alpha}{(1-\alpha)} L/K}\right]^{-1} = 1 \tag{2.20}$$
which is indeed constant. This is illustrated in Figure 2. Points $\mathbf{e_1}$ and $\mathbf{e_2}$ are two different points on the same isoquant $K^\alpha L^{1-\alpha} = \overline{Y}$ of a Cobb-Douglas production function. The slope of the isoquant at points $\mathbf{e_1}$ and $\mathbf{e_2}$ is given by minus the tangent of the angles A and B respectively. The labour-capital ratio L/K at points $\mathbf{e_1}$ and $\mathbf{e_2}$ is the slope of the chord from the origin to the point, which is the tangent of the angles C and D respectively. Heuristically, the elasticity

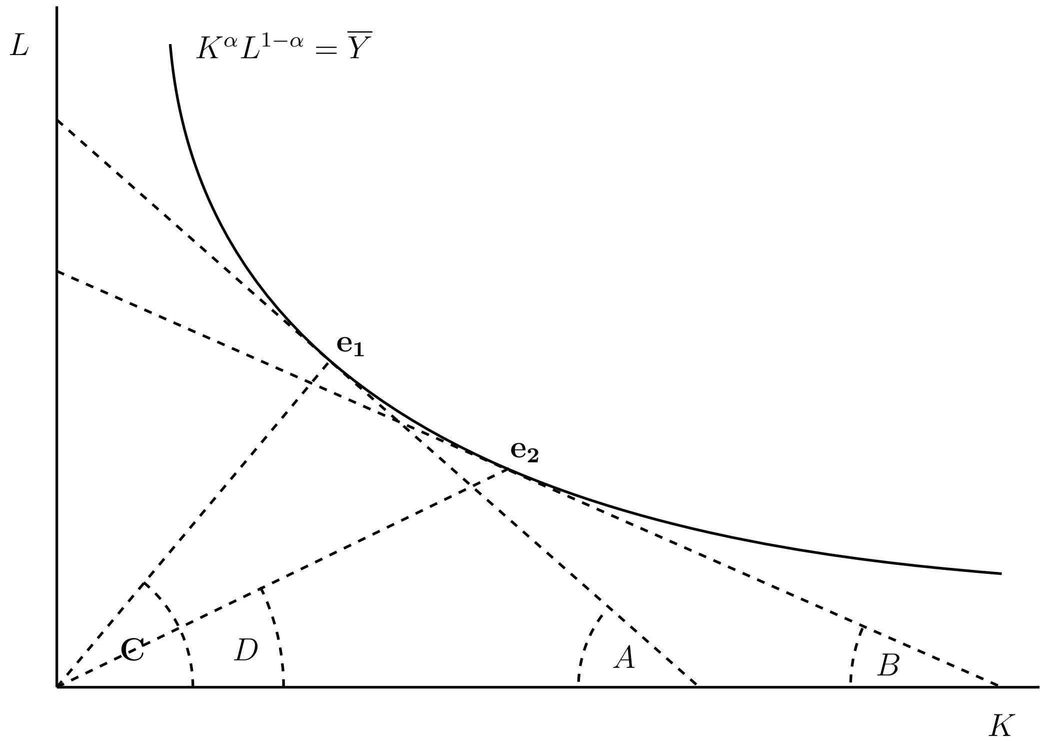

Figure 2: Cobb-Douglas isoquant: $K^\alpha L^{1-\alpha} = \overline{Y}$

of substitution is the ratio of the change in the angle C to D to the change in the angle A to B. With the Cobb-Douglas production function, this elasticity of substitution is the same at all points on the isoquant and for all isoquants and is equal to unity. With more curved (convex) isoquants, the elasticity of substitution would be lower and with less convex isoquants higher.

The Cobb-Douglas is a special member of the class of *Constant Elasticity of Substitution* (*CES*) production functions of Arrow *et al.* (1961) defined by

$$F(K,L) = \big(\alpha(bK)^\psi + (1-\alpha)[(1-b)L]^\psi\big)^{1/\psi} \tag{2.21}$$

where $0 < \alpha < 1$, $0 < b < 1$ and $\psi < 1$. The elasticity of substitution between capital and labour in the *CES* production function is given by $1/(1-\psi)$. It can be shown that the Cobb-Douglas production function is a limiting case of the *CES* production function where $\psi \to 0$.

Another limiting case of the *CES* production function is where $\psi \to -\infty$ so that the elasticity of substitution $1/(1-\psi) \to 0$. In this case the production function reduces to

$$F(K,L) = \min(cK, dL) \tag{2.22}$$

where $c > 0$ and $d > 0$ are constants. This is the Leontief production function in which labour and capital can only be used in fixed proportions d/c so that there can be no substitution between them. An example would be a factory in which each worker operates a single machine. Increasing the number of workers without increasing the number of machines (or *vice versa*) will not increase output. Output can only be increased by increasing both labour and capital in fixed proportions (in this case one-to-one). Figure 3 illustrates an isoquant of the Leontief production

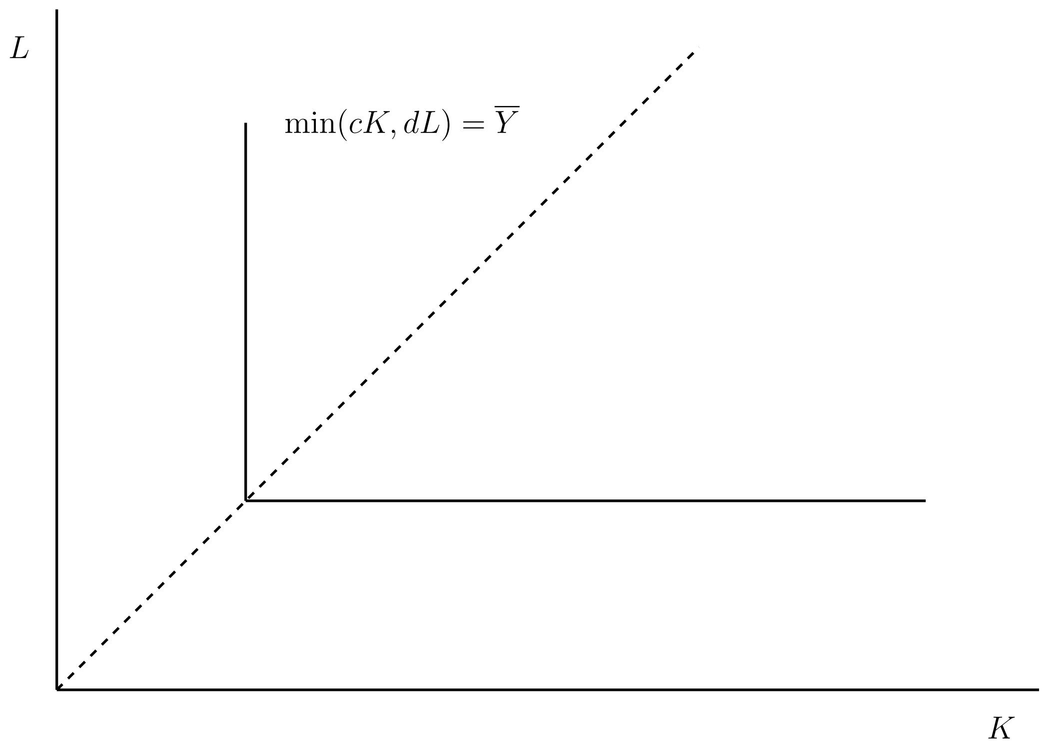

Figure 3: Leontief isoquant: $\min(cK, dL) = \overline{Y}$

function. The slope of the dotted chord from the origin is the labour-capital ratio d/c and all isoquants of the production function minimise factor inputs along this same chord.

3 A note on growth rates

We use the notation $\dot{x}$ to represent the time derivative of a variable x, dx/dt. The growth rate of x is defined by

$$\frac{1}{x}\frac{dx}{dt} = \frac{\dot{x}}{x}. \tag{3.1}$$

Note that if $x = y \cdot z$ then, by the rule for the derivative of a product,

$$\frac{dx}{dt} = z\frac{dy}{dt} + y\frac{dz}{dt} \tag{3.2}$$

and, dividing by x,

$$\frac{1}{x}\frac{dx}{dt} = \frac{z}{x}\frac{dy}{dt} + \frac{y}{x}\frac{dz}{dt} \tag{3.3}$$

or

$$\frac{1}{x}\frac{dx}{dt} = \frac{1}{y}\frac{dy}{dt} + \frac{1}{z}\frac{dz}{dt} \tag{3.4}$$

which, in dot notation, is

$$\frac{\dot{x}}{x} = \frac{\dot{y}}{y} + \frac{\dot{z}}{z}. \tag{3.5}$$

This useful result that the growth rate of the product of two variables is equal to the sum of the growth rates of the two variables, generalises to the case of more than two variables.

4 Growth of the factors of production

In the production function (2.1), both output Y and the factors of production K and L are assumed to be functions of time. Output changes over time as a result of changes in the factors of production. The capital stock in the economy K is assumed to depreciate at a constant rate δ ($0 < \delta < 1$) as it wears out. Without some capital investment, it would continually decline. However, the capital stock can be increased by investment so that the stock of capital changes over time according to the differential equation

$$\dot{K} = I - \delta K \tag{4.1}$$

where I is the rate of (gross) capital investment (which is a flow over time).

What determines the rate of investment? Total output in the economy, Y, is equal to total expenditure which (in this closed economy assuming no exports or imports and no government) is the sum of consumption expenditure C and investment expenditure I as defined by the expenditure identity

$$Y \equiv C + I. \tag{4.2}$$

Output is also equal to total income which is the sum of consumption C and savings S as defined by the income identity

$$Y \equiv C + S. \tag{4.3}$$

It follows from (4.2) and (4.3) that investment is equal to savings, i.e.

$$I \equiv S. \tag{4.4}$$

The savings assumption of the Solow growth model is that savings S are exogenously determined and are assumed to be simply a constant proportion s of income:

$$S = sY \tag{4.5}$$

where $0 < s < 1$. Since investment is equal to savings it follows that

$$I = sY = sF(K, L) \tag{4.6}$$

and from (4.1)

$$\dot{K} = sY - \delta K. \tag{4.7}$$

Finally, it is assumed that the labour force L grows at a constant rate n,

$$\frac{\dot{L}}{L} = n \quad \text{or} \quad \dot{L} = nL. \tag{4.8}$$

5 The Balanced Growth Path

Having made the assumptions (4.5) and (4.8) on the dynamics of the factors of production, we can now derive the key equation of the Solow growth model. Since

$$k = K/L \tag{5.1}$$

it follows from the rule for the derivative of a ratio that

$$\frac{dk}{dt} = \frac{1}{L}\frac{dK}{dt} - \frac{K}{L}\frac{1}{L}\frac{dL}{dt} \tag{5.2}$$

or, using dot notation,

$$\dot{k} = \frac{\dot{K}}{L} - k\frac{\dot{L}}{L}. \tag{5.3}$$

Dividing (4.7) by L we get

$$\frac{\dot{K}}{L} = s\frac{Y}{L} - \delta\frac{K}{L} \tag{5.4}$$

or

$$\frac{\dot{K}}{L} = sy - \delta k \tag{5.5}$$

and, substituting (5.5) and (4.8) into (5.3), we derive the fundamental growth equation

$$\dot{k} = sy - \delta k - nk \tag{5.6}$$

or, using (2.7),

$$\dot{k} = sf(k) - (n + \delta)k. \tag{5.7}$$

This equation says that the rate of change of capital per worker is the difference between two terms. The first is investment per worker I/L while the second is break-even investment which is the amount of investment per worker needed just to keep capital per worker constant.

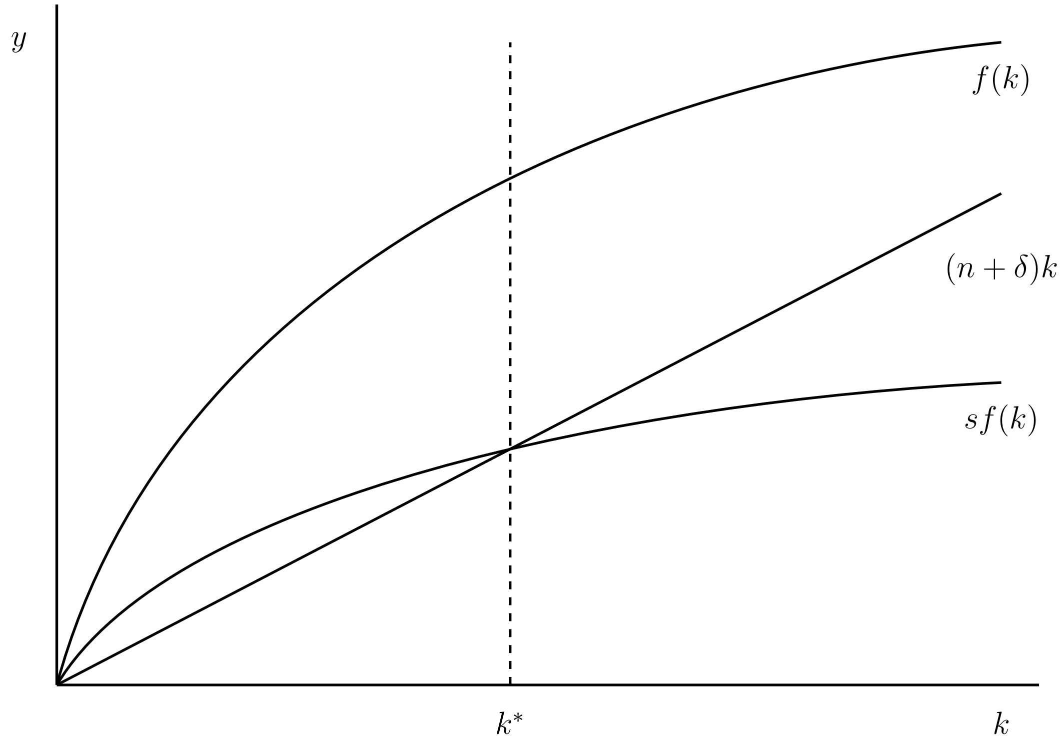

Figure 4: The balanced growth path: $sf(k^*) = (n + \delta)k^*$

The second term in (5.7) is linear in k and is increasing as long as $(n+\delta) > 0$ which is a plausible assumption. However, the first term is non-linear since from (2.11) $f''(k) < 0$ and, from the Inada conditions, we know that $f'(k)$ decreases from ∞ at $k = 0$ to 0 at $k = \infty$. It follows that when k is small, the first term will be larger than the second term and so $\dot{k}$ will be positive and k will be increasing. However, as k increases $f'(k)$ decreases and there will come a point when the second term outweighs the first and beyond this $\dot{k}$ will become negative so that k starts to decrease. The equilibrium point is where $\dot{k} = 0$. Let us denote this value of k as k^*. For $k > 0$, this equilibrium is a unique one and, as long as $(n+\delta) > 0$, it is a stable equilibrium since, if k is above or below k^*, k will adjust to bring it back to k^*. The equilibrium is illustrated in Figure 4.

The equilibrium point defines the *balanced growth path.* At this point k will be constant at k^* but the capital stock itself

$$K = Lk \tag{5.8}$$

will, (using 3.5), be growing at the rate

$$\frac{\dot{K}}{K} = \frac{\dot{L}}{L} + \frac{\dot{k}}{k} = n. \tag{5.9}$$

Similarly, although in equilibrium output per worker will be constant at $f(k^*)$, output itself,

$$Y = Lf(k^*), \tag{5.10}$$

will also be growing at the rate n.

6 Savings and the Golden Rule of Accumulation

How can a country attempt to change its rate of growth? One possibility is to alter the savings rate s. Increasing the savings rate will lead to a move to a new steady state with higher output per worker y. However, the effect on consumption per worker $c = C/L$ is not clear. Dividing the income identity (4.3) by Y we have

$$\frac{Y}{Y} = \frac{C}{Y} + \frac{S}{Y} \tag{6.1}$$

or

$$s = 1 - \frac{C}{Y} \tag{6.2}$$

and, substituting into (5.7),

$$\dot{k} = (1 - \frac{C}{Y})f(k) - (n + \delta)k = 0 \tag{6.3}$$

in steady state. Rearranging, and using * to denote steady state values

$$\frac{C^*}{Y^*}f(k^*) = f(k^*) - (n + \delta)k^* \tag{6.4}$$

but

$$\frac{C^*}{Y^*}f(k^*) = \frac{C^*}{Y^*}\frac{Y^*}{L^*} = c^* \tag{6.5}$$

so we have

$$c^*(s) = f(k^*(s)) - (n + \delta)k^*(s) \tag{6.6}$$

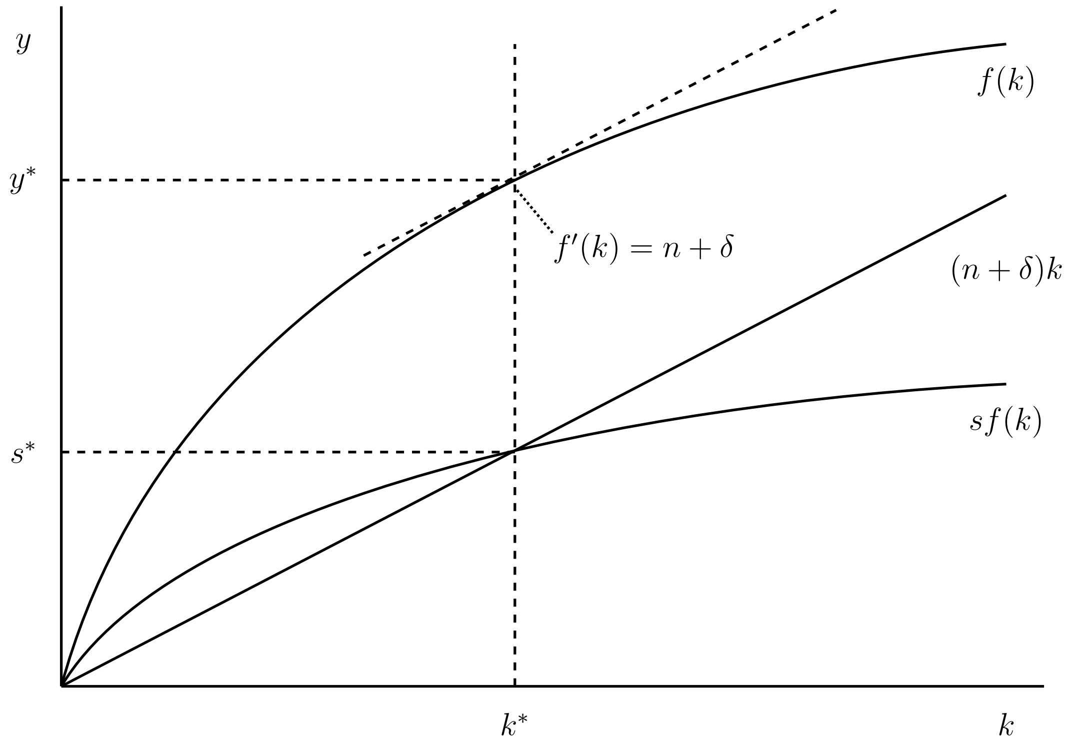

Figure 5: The golden rate of savings: $f'(k^*(s)) = n + \delta$

where we have made explicit the dependency of the steady state capital per worker k^* and the steady state consumption per worker c^* on the savings rate s. This equation is known as the *golden rule of capital accumulation*. It defines the relationship between the steady-state consumption per worker c^* and the savings rate s and this relationship is non-linear. For small values of k^*, c^* is increasing in k^* but for high values c^* is decreasing in k^*. c^* is maximised by setting dc/ds to zero which implies

$$(f'(k^*(s)) - (n + \delta))\frac{dk^*}{ds} = 0. \tag{6.7}$$

Since $dk^*/ds > 0$, this implies

$$f'(k^*(s)) = n + \delta. \tag{6.8}$$

The savings rate, s^*, that satisfies this equation and maximises consumption per worker is known as the *golden rate of savings*. It is illustrated in Figure 5. At the equilbrium point the slope of the production function $f(k)$ is equal to the slope of the line $(n + \delta)k$. The golden rate of savings is s^* and consumption per head c^* is equal to the distance y^* minus s^*.

Is the golden rate of savings the *optimal* rate? This isn't clear as the model doesn't contain an explicit utility or welfare function. Savings can be thought of as a way of foregoing consumption today in order to consume tomorrow. If the current savings rate is above the golden rate, then reducing the savings rate to move to the golden rate wil increase per capita consumption today and in the future and this should unambiguously increase welfare (assuming of course that welfare is an increasing function of consumption per head). However, if the current rate is below the golden rate then increasing savings to move to the golden rate involves sacrificing consumption per capita today in order to achieve higher consumption per capita in the future. Whether this increases or decreases welfare will depend on the rate of time discount in the welfare function and the Solow model does not address this question. In a later lecture we will look at some models that do make explicit assumptions about inter-temporal welfare and so can define an optimal time path for consumption and savings.

7 Conclusions

The simplest form of the Solow-Swan neo-classical growth model considered here has some strong implications. Assuming an exogenous rate of savings, the model shows that a steady state equilbrium rate of growth exists in which output and capital grow at the same rate as the labour force. An economy not growing at this steady state will naturally adjust its rate of growth until it reaches the steady state rate. This gives an explantion for the fast growth rates of developing economies with high population growth rates who in this interpretation are still adjusting towards a steady state growth rate consistent with their rate of labour growth. Furthermore, the model implies that there will exist a golden rate of savings which defines a steady state growth rate that maximises consumption per head in the economy, although to reach this golden rate may involve a sacrifice of current consumption.

However, another conclusion of this simple model is that, in steady state, output per worker will be constant. The model cannot explain the persistent increases in per capita income enjoyed by many developed economies (which could reasonably be assumed to have achieved steady state growth). In order to overcome this limitation, we need to introduce into the model some mechanism whereby output per worker can be allowed to grow persistently. This is the purpose of the next lecture.

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