Macroeconomics
Lecture 2: The Solow Growth Model with Technical Progress

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1 Introduction

In last week's lecture we considered the basic Solow-Swan growth model (Solow (1956), Swan (1956)). In this model, in steady-state, output and capital grow at the rate of growth of the labour force and so the model cannot explain the steady state growth of output per capita that has been observed in most economies. To explain this we need to introduce into the model some autonomous process that causes the production function to shift up over time so that output can grow over time even when the labour and capital inputs are not growing. This we call technical progress.

2 Technical Progress

Technical progress is a continuing process that increases the efficiency of the production process over time, allowing more output to be produced from the same quantities of factor inputs, capital K and labour L. In the real world, technical progress will depend on the continued generation of new discoveries and inventions. Some of these might come about by chance (serendipitous) but most will be the result of focused research and development carried out in universites or in corporate or goverment laboratories. This research needs to be funded somehow and so the rate of technical progress would be expected to depend on the investment in R&D which is likely to vary according to economic conditions. However, in this simple model, we will assume that technical progress is an exogenous process defined by

$$A(t) \tag{2.1}$$

growing at a constant rate

$$\frac{\dot{A}}{A} = g. \tag{2.2}$$

How does technical progress affect the production process? There are three possibilities: technology may be labour-saving, allowing output to be produced with less labour input, it may be capital-saving, using less capital input, or it may save both capital and labour. Three different concepts of neutral technical progress have been proposed in the literature by Hicks (1932), Harrod (1942) and Solow (1969).

2.1 Hicks-neutral Technical Progress

Hicks (1932) defined a technological innovation to be neutral (*Hicks-neutral*) if the ratio of the factor marginal products remains unchanged for a given capital-labour ratio. The Hicks-neutral production function can be written as

$$Y(t) = A(t)F(K(t), L(t)) \tag{2.3}$$

and it can be seen that the ratio of factor marginal products

$$\frac{\partial Y/\partial K}{\partial Y/\partial L} = \frac{AF_K}{AF_L} = \frac{F_K}{F_L} \tag{2.4}$$

does not depend on A.

2.2 Harrod-neutral Technical Progress

Harrod (1942) defined a technological innovation to be neutral (*Harrod-neutral*) if the relative input shares

$$\frac{KF_K}{LF_L} \tag{2.5}$$

are unchanged for a given capital-output ratio. The Harrod-neutral production function can be written as

$$Y(t) = F(K(t), A(t)L(t)). \tag{2.6}$$

This form of technical progress is called *labour-augmenting* because it acts to increase output in the same way as an increase in the stock of labour.

2.3 Solow-neutral Technical Progress

Solow (1969) defined a technological innovation to be neutral (*Solow-neutral*) if the relative input shares (2.5) remain unchanged for a given labour-output ratio. The Solow-neutral production function can be written as

$$Y(t) = F(A(t)K(t), L(t)). \tag{2.7}$$

This form of technical progress is called *capital-augmenting* because it acts to increase output in the same way as an increase in the stock of capital.

3 The Solow Model with Technical Progress

It turns out that the only form of exogenous technical progress that is consistent with the existence of a steady state growth path is *labour-augmenting* or *Harrod-neutral* technical progress. Thus in the extended version of the Solow model including technical progress, we assume a production function of the form (now dropping the time argument (t))

$$Y = F(K, AL). \tag{3.1}$$

AL is now the labour input measured in efficiency units. Constant returns to scale implies

$$\lambda Y = F(\lambda K, \lambda AL) \tag{3.2}$$

and setting $\lambda = 1/(AL)$ we have the intensive production function

$$\frac{Y}{AL} = F(\frac{K}{AL}, 1) \tag{3.3}$$

or

$$\hat{y} = f(\hat{k}) \tag{3.4}$$

where $\hat{y} = Y/(AL) = y/A$ is output per effective unit of labour and $\hat{k} = K/(AL) = k/A$ is capital per effective unit of labour. The capital-labour ratio K/L is now given by $A\hat{k}$.

We assume that

$$f'(\hat{k}) > 0 \tag{3.5}$$

and

$$f''(\hat{k}) < 0 \tag{3.6}$$

as well as the Inada conditions (Inada (1964)):

$$\lim_{\hat{k}\to 0} f'(\hat{k}) = \infty \quad \text{and} \quad \lim_{\hat{k}\to\infty} f'(\hat{k}) = 0. \tag{3.7}$$

The constant returns to scale Cobb-Douglas production function with labour-augmenting technical progress takes the form

$$F(K, AL) = K^{\alpha}(AL)^{1-\alpha} \tag{3.8}$$

with $0 < \alpha < 1$ and in intensive form is given by

$$f(\hat{k}) = F(K/(AL), 1) = (K/(AL))^{\alpha} = \hat{k}^{\alpha}. \tag{3.9}$$

As in the basic model, we assume that capital changes according to

$$\dot{K} = I - \delta K \tag{3.10}$$

where investment I is equal to savings S which is a constant proportion s of income:

$$S = sY \tag{3.11}$$

for $0 < s < 1$. It follows that (3.10) can be written as

$$\dot{K} = sY - \delta K. \tag{3.12}$$

The labour force L grows at a constant rate n,

$$\frac{\dot{L}}{L} = n \tag{3.13}$$

which means that effective labour AL grows at the rate

$$\frac{\dot{AL}}{AL} = \frac{\dot{A}}{A} + \frac{\dot{L}}{L} = g + n. \tag{3.14}$$

4 The Balanced Growth Path

Using the assumptions (3.11) and (3.13) and (2.2) we can now derive the key equation of the Solow growth model with technical progress. Since

$$\hat{k} = K/(AL) \tag{4.1}$$

it follows from the rule for the derivative of a ratio that

$$\frac{d\hat{k}}{dt} = \frac{1}{AL}\frac{dK}{dt} - \frac{K}{AL}\frac{1}{AL}\frac{d(AL)}{dt} \tag{4.2}$$

or, using dot notation,

$$\dot{\hat{k}} = \frac{\dot{K}}{AL} - \hat{k}\frac{\dot{AL}}{AL}. \tag{4.3}$$

Dividing (3.12) by AL we get

$$\frac{\dot{K}}{AL} = s\frac{Y}{AL} - \delta\frac{K}{AL} \tag{4.4}$$

or

$$\frac{\dot{K}}{AL} = s\hat{y} - \delta\hat{k} \tag{4.5}$$

and, substituting (4.5) and (3.14) into (4.3), we derive the fundamental growth equation

$$\dot{\hat{k}} = s\hat{y} - \delta\hat{k} - (g + n)\hat{k} \tag{4.6}$$

or, using (3.4),

$$\dot{\hat{k}} = sf(\hat{k}) - (g + n + \delta)\hat{k}. \quad (4.7)$$

This equation now says that the rate of change of capital per unit of effective labour is the difference between investment per unit of effective labour $I/(AL)$ and break-even investment which is the amount of investment per unit of effective labour needed just to keep capital per unit of effective labour constant. The equilibrium level of capital per unit of effective labour, $\hat{k}^*$, is illustrated in Figure 1.

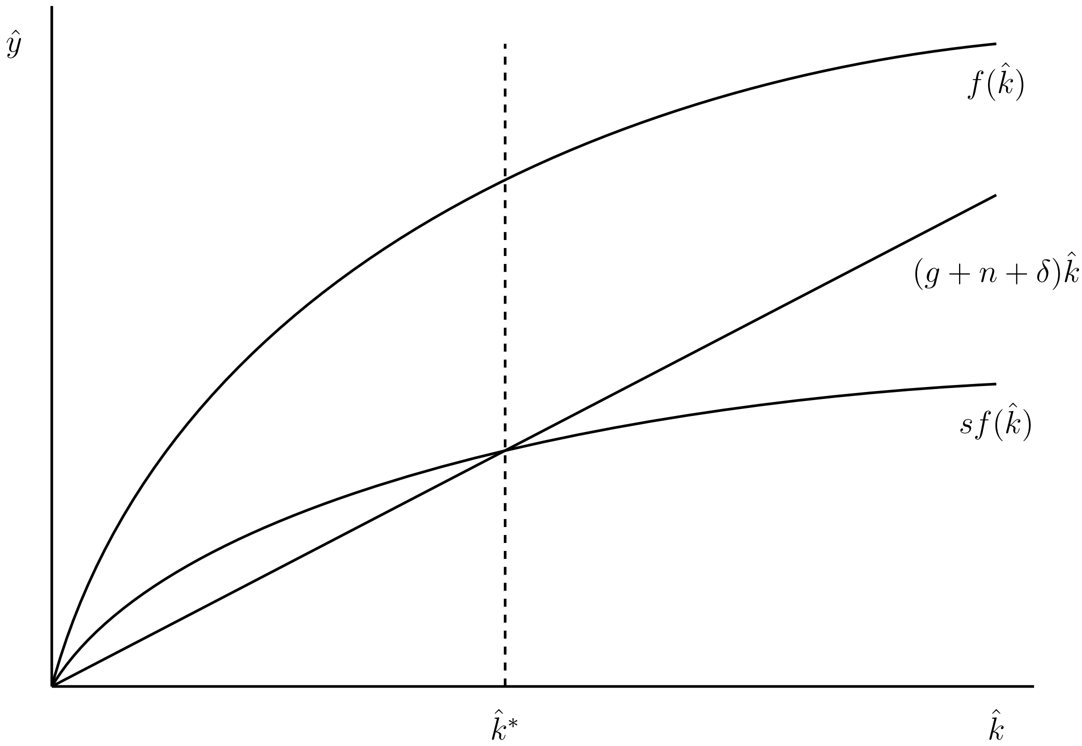

Figure 1: The balanced growth path: $sf(\hat{k}^*) = (g + n + \delta)\hat{k}^*$

On the balanced growth path when $\dot{\hat{k}} = 0$, $\hat{k}$ will be constant at $\hat{k}^*$ but capital per capita

$$\frac{K}{L} = A\hat{k} \quad (4.8)$$

will be growing at the rate

$$\frac{\dot{K/L}}{K/L} = \frac{\dot{A}}{A} + \frac{\dot{\hat{k}}}{\hat{k}} = g \quad (4.9)$$

and the capital stock itself

$$K = (AL)\hat{k} \tag{4.10}$$

will be growing at the rate

$$\frac{\dot{K}}{K} = \frac{\dot{AL}}{AL} + \frac{\dot{\hat{k}}}{\hat{k}} = g + n. \tag{4.11}$$

Similarly, although, in equilibrium, output per effective unit of labour will be constant at $f(\hat{k}^*)$, output itself,

$$Y = (AL)f(\hat{k}^*) \tag{4.12}$$

will also be growing at the rate $g + n$ and output per worker at the rate g. Thus technical progress does deliver a steady state growth rate for per capita output equal to the growth rate of technical progress itself.

Since capital and output both grow at the same steady state rate $g + n$, it follows that the model predicts that the capital-output ratio K/Y will be constant. On the other hand since labour only increases at the rate n, the capital-labour ratio would be expected to increase.

5 Returns to the Factors of Production

If we assume perfect competition, then the rates of return to the factors of production, capital K and labour L should be equal to their marginal products. Denoting the rate of return to capital, the *rental rate*, as r and rate of return to labour, the *wage rate*, as w, we have

$$r = \frac{\partial F(K, AL)}{\partial K} \tag{5.1}$$

and

$$w = \frac{\partial F(K, AL)}{\partial L}. \tag{5.2}$$

The shares of income going to capital and labour are given by

$$\frac{rK}{Y} = \frac{\partial F(K, AL)}{\partial K}\frac{K}{Y} \tag{5.3}$$

and

$$\frac{wL}{Y} = \frac{\partial F(K, AL)}{\partial L}\frac{L}{Y} \tag{5.4}$$

respectively, which are the elasticities of output to factor inputs K and L.

For a production function $F(K, AL)$ with constant returns to scale, Euler's theorem for homogeneous functions (of degree one) states that

$$F(K, AL) = \frac{\partial F(K, AL)}{\partial K} K + \frac{\partial F(K, AL)}{\partial L} L \tag{5.5}$$

and so

$$\frac{F(K, AL)}{Y} = \frac{\partial F(K, AL)}{\partial K} \frac{K}{Y} + \frac{\partial F(K, AL)}{\partial L} \frac{L}{Y} = 1 \tag{5.6}$$

or, from (5.3) and (5.4),

$$\frac{rK}{Y} + \frac{wL}{Y} = 1 \tag{5.7}$$

so that the shares of output going to the factor inputs sum to unity and there are no *pure profits*.

In steady state growth in the Solow-Swan model, the capital output ratio K/Y and the output per unit of labour $Y/L = y$ are both constant. This means that the factor input shares will also be constant. However, when the economy is not on the steady state balanced growth path, the ratio of capital to labour k is changing so that the marginal factor products are changing as the economy moves along the production function.

For the Cobb-Douglas case

$$F(K, AL) = K^{\alpha}(AL)^{1-\alpha} = K^{\alpha}A^{1-\alpha}L^{1-\alpha} \tag{5.8}$$

the marginal factor products are

$$\frac{\partial F(K, AL)}{\partial K} = \alpha K^{\alpha-1}(AL)^{1-\alpha} = \alpha \frac{Y}{K} \tag{5.9}$$

and

$$\frac{\partial F(K, AL)}{\partial L} = (1-\alpha)K^{\alpha}A^{1-\alpha}L^{-\alpha} = (1-\alpha)\frac{Y}{L} \tag{5.10}$$

so, substituting into (5.5), gives

$$F(K, AL) = \alpha Y + (1-\alpha)Y \tag{5.11}$$

so that the factor shares are α and $1-\alpha$ respectively. For this case, the factor shares are constant at all points on the production function, not just on the steady state balanced growth path. This is a unique property of the Cobb-Douglas production function.

6 Fitting the Facts

How do the predictions of the Solow model fit the facts about growth in the real world? In Kaldor (1963), the British economist Nicholas (later Lord) Kaldor made a list of six stylised facts about growth in a single (industrialised) country.

1. Per capita output grows over time, and its growth rate does not tend to diminish.
2. Physical capital per worker grows over time.
3. The rate of return to capital is nearly constant.
4. The ratio of physical capital to output is nearly constant.
5. The shares of labour and physical capital in national income are nearly constant.
6. The growth rate of output per worker differs substantially across countries.

The Solow-Swan model with technical progress predicts that per capita output y in steady state and capital per worker k both grow at the constant rate g. It also predicts that the capital output ratio is constant. On the balanced growth path, it predicts that both the rate of return to capital r and that the income shares of capital and effective labour will be constant, although off the balanced growth path they may be changing (depending on the form of the production function). Thus the predictions of the model are consistent with the first five of Kaldor's stylised facts.

On the final stylised fact, the observed differences between growth rates in output per worker in different countries, the predictions of the Solow-Swan model depend on whether the countries are assumed to be already on a balanced growth path or whether they are still adjusting towards one. On the balanced growth path, output per worker in all countries will be growing at the rate g, the growth rate of technical progress. If this rate were the same for all countries, then all countries on a balanced growth path would be growing at the same per capita rate. However, it might be that technological growth rates differ between countries. Since the model does not explain why technical progress occurs but treats it as exogenous, it is not clear whether the growth rate could be different in different countries or whether it would always converge, as all countries adopted new technical innovations.

If countries have not yet reached a balanced growth path, then the Solow-Swan model predicts that in countries below the equilibrium $\hat{k}^*$, which is to say countries with low y, output per worker will be growing faster than g while in countries above the equilibrium $\hat{k}^*$, which is to say countries with high y, output per worker will be growing slower than g. Thus when countries are still adjusting to an equilibrium growth rate, the model predicts that poor countries will be growing faster than rich countries.

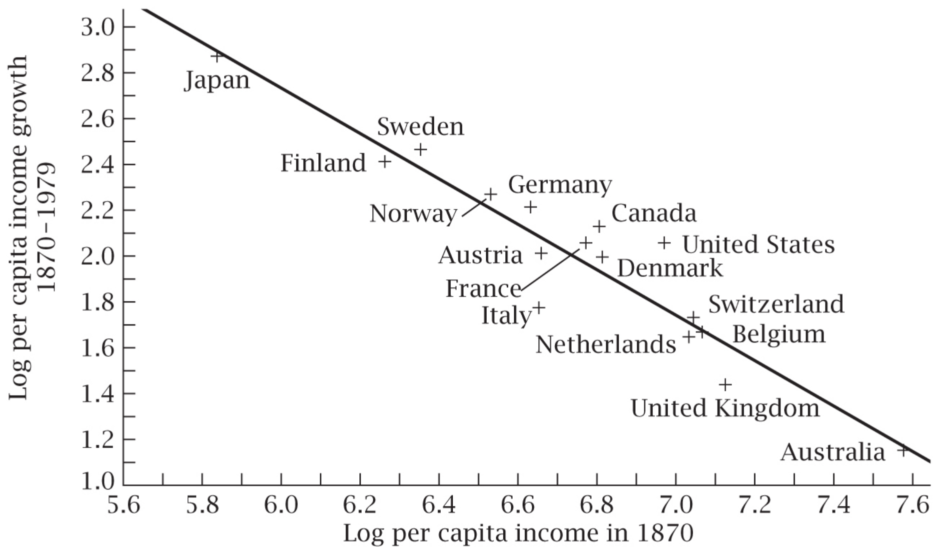

FIGURE 1.7 Initial income and subsequent growth in Baumol's sample (from DeLong, 1988; used with permission)

Figure 2: [Figure 1.7 in Romer (2012)]

The issue of whether in fact poor countries tend to grow faster than rich countries (and therefore tend to catch up over time in terms of per capita output) is one that has long interested economists. It is known as the issue of *convergence*. Baumol (1986) analysed convergence between 1870 and 1979 in 16 industrialised countries and his results are illustrated in Figure 2 (which is reproduced from Romer (2012)). The figure plots log per capita growth over the period 1870 to 1979 against initial log per capita income in 1870. The points seem to lie on a downward sloping line showing that countries with a lower initial per capita income (like Japan) grew faster over the period than countries with a higher initial per capita income (like the UK and Australia). This result seems to support convergence. However, DeLong (1988) argues that Baumol's result is largely spurious since his sample concentrates on countries that industrialised early (and so have a long data series) or that have grown rapidly since, which biases the result. DeLong expands the sample and finds a much weaker result. Romer (2012) updates the analysis

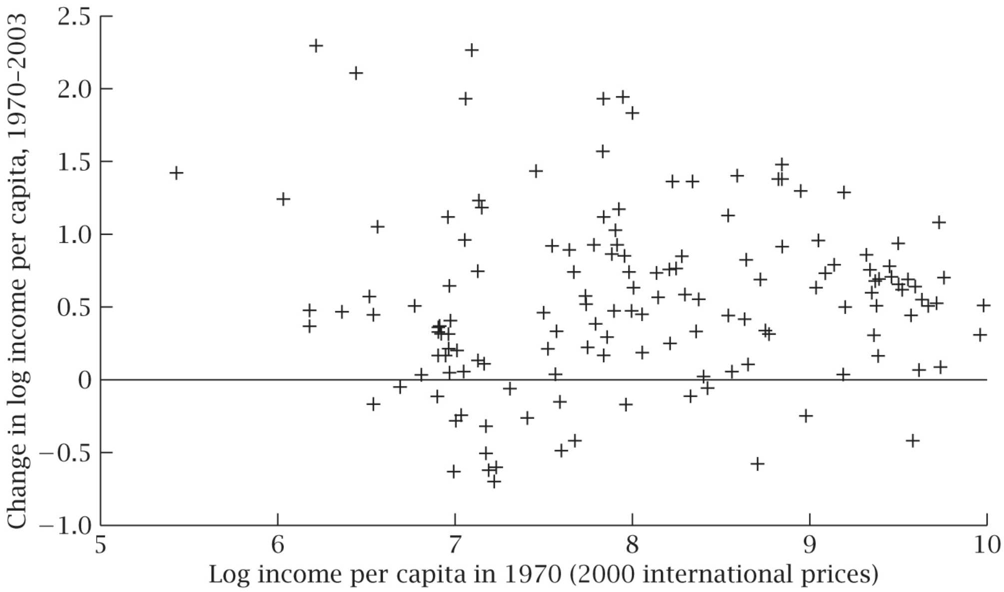

FIGURE 1.9 Initial income and subsequent growth in a large sample

Figure 3: [Figure 1.9 in Romer (2012)]

to consider growth over the shorter period 1970 to 2003 for a much larger sample that includes most of the non-communist world. Romer's results are illustrated in Figure 3 which shows no clear evidence of convergence and seems, rather, a random scatterplot. Thus the issue of convergence is still a controversial one.

7 Conclusions

The version of the Solow-Swan neo-classical growth model allowing for labour-augmenting technical progress explains most of the stylised facts about growth. However, it does not explain why technical progress actually occurs, whether it could be different in different countries or whether it can be expected to continue indefinitely. This is rather worrying given that in the model it is the sole determinant of sustained per capita growth. In a later lecture we will be looking at endogenous growth models which provide mechanisms to explain technological growth.

Another assumption of the model is that the sole factors of production are capital and labour and that these are in potentially infinite supply. In the next lecture we will look at the introducton of additional factors of production, land

and natural resources, that are either in fixed supply or are running out.

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