Macroeconomics
Lecture 4: Endogenous Growth

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1 Introduction

The Solow-Swan growth model with technical progress successfully accounts for growth in output per head. However, the model does not *explain* why technical progress occurs but simply treats it as exogenous. It also cannot explain the variations in per capita growth rates between different countries.

In this lecture we look at several models that attempt to explain technical progress in different ways, thereby making growth an endogenous process, depending on the factors of production K and L. In general, we can write

$$A = A(K, L) \tag{1.1}$$

so that the conventional neo-classical production function

$$Y = F(K, AL) \tag{1.2}$$

becomes

$$Y = F(K, A(K, L)L) = \bar{F}(K, L) \tag{1.3}$$

but where $\bar{F}(K, L)$ may display *increasing* returns to scale at the aggregate level. Paul Romer's (1994) paper is a good survey of endogenous growth models.

2 Solow Models With Human Capital

Before we turn to explicit models of technical progress, we look at an important extension to the Solow-Swan model that allows for investment in the quality of the labour force. In the standard Solow-Swan model, labour is a homogeneous input and the only form of investment is investment in physical capital. In human capital models the quality of the labour force can be improved by investment in education or training.

2.1 Mankiw Romer and Weill

Mankiw, Romer and Weill (1992) consider a model which includes two separate forms of capital: physical capital K and human capital H. Assuming a Cobb-Douglas production function we have

$$Y = K^{\alpha} H^{\beta} (AL)^{1-\alpha-\beta} \tag{2.1}$$

where H is the stock of human capital, $\alpha > 0$, $\beta > 0$ and $\alpha + \beta < 1$. Note that in this model human capital H is a separate factor of production from normal labour L. It can be thought of as skilled as opposed to unskilled labour.

Both physical capital K and human capital H can be increased by investment but are assumed to depreciate over time at the same rate δ. Investment in human capital can be thought of as education and training and the depreciation of human capital as a skill deterioration (a sort of *rustiness*) that comes about if education and training are not continually refreshed.

K and H change over time according to the differential equations

$$\dot{K} = I_k - \delta K \tag{2.2}$$

and

$$\dot{H} = I_h - \delta H \tag{2.3}$$

where now I_k is investment in physical capital and I_h is investment in human capital and

$$I_k + I_h = I = S. \tag{2.4}$$

Some fraction of savings goes towards investment in human capital and the rest goes to investment in physical capital. Specifically,

$$I_k = s_k Y \tag{2.5}$$

and

$$I_h = s_h Y \tag{2.6}$$

where $0 < s_k < 1$, $0 < s_h < 1$ and $s_k + s_h = s < 1$.

Substituting (2.5) and (2.6) into (2.2) and (2.3) and dividing by AL gives

$$\frac{\dot{K}}{AL} = s_k \frac{Y}{AL} - \delta \frac{K}{AL} \tag{2.7}$$

and

$$\frac{\dot{H}}{AL} = s_h \frac{Y}{AL} - \delta \frac{H}{AL}. \tag{2.8}$$

Defining

$$\hat{y} = \frac{Y}{AL}\ , \qquad \hat{k} = \frac{K}{AL} \qquad \text{and} \qquad \hat{h} = \frac{H}{AL}$$

and noting that

$$\dot{\hat{k}} = \frac{\dot{K}}{AL} - \hat{k}\frac{(\dot{AL})}{AL}\ , \qquad \dot{\hat{h}} = \frac{\dot{H}}{AL} - \hat{h}\frac{(\dot{AL})}{AL} \qquad \text{and} \qquad \frac{(\dot{AL})}{AL} = n + g,$$

equations (2.7) and (2.8) can be rewritten as

$$\dot{\hat{k}} = s_k\hat{y} - (n + g + \delta)\hat{k} \tag{2.9}$$

and

$$\dot{\hat{h}} = s_h\hat{y} - (n + g + \delta)\hat{h}. \tag{2.10}$$

Equations (2.9) and (2.10) jointly define the steady-state growth path in this model. For balanced growth, both $\dot{\hat{k}} = 0$ and $\dot{\hat{h}} = 0$ so that

$$\hat{y}^* = \frac{n + g + \delta}{s_k}\hat{k}^* \tag{2.11}$$

and

$$\hat{y}^* = \frac{n + g + \delta}{s_h}\hat{h}^*. \tag{2.12}$$

In steady state Y, K and H all grow at the same rate $n + g$ which is the sum of the growth rate of the labour force n and the growth rate of technical progress g.

The addition of human capital to the Solow model in Mankiw, Romer and Weill (1992) was important because it helped solve an empirical puzzle in the standard Solow model. Estimation of the parameter α in the Cobb-Douglas production function

$$Y = K^{\alpha}(AL)^{1-\alpha} \tag{2.13}$$

had consistently produced values for $\hat{\alpha}$ of approximately 1/3 for most countries, implying that the share of output going to labour is approximately 2/3. However, this value of α is far too low for observed differences in savings rates to account for the variations in output per capita that we see in different countries. Recall Exercise 1 where you derived the result that for the Cobb-Douglas production function in the standard Solow model (in the absence of technical progress)

$$y^* = \left(\frac{s}{n + \delta}\right)^{\alpha/(1-\alpha)}.$$

With $\alpha = 1/3$, this means that

$$y^* \propto s^{0.5}$$

so that the savings rate would need to quadrupal for output per head to double.

In the model with human capital the steady-state output per unit of effective labour $\hat{y}$ is given by

$$\hat{y}^* = \left(\frac{s_k^\alpha s_h^\beta}{(n+g+\delta)^{\alpha+\beta}} \right)^{\frac{1}{1-\alpha-\beta}} \tag{2.14}$$

and typical estimates of α and β are $\alpha = \beta = 1/3$. These estimates mean that

$$\hat{y}^* \propto s_k s_h$$

so that if both s_k and s_h double, then output per (effective) worker will quadrupal. This is much more in line with the empirical evidence.

2.2 A Simple Human Capital Model

A simpler human capital model is given by

$$Y = K^\alpha (AH)^{1-\alpha} \tag{2.15}$$

where $0 < \alpha < 1$ and H is now the total amount of productive services supplied by workers, which follows the equation

$$H = e^{\phi\mu} L \tag{2.16}$$

where L is the number of workers, μ is the proportion of time a worker spends on education and training and ϕ is a parameter with $\phi > 0$. The standard assumptions of the Solow model with technical progress:

$$\dot{K} = sY - \delta K \ , \qquad \frac{\dot{L}}{L} = n \qquad \text{and} \qquad \frac{\dot{A}}{A} = g$$

are all assumed to hold and the dynamics of this model are exactly the same as the Solow model so that in balanced growth

$$\dot{\hat{k}} = s\hat{y} - (n+g+\delta)\hat{k} = 0$$

where now $\hat{k} = K/(AH)$ and $\hat{y} = Y/(AH)$. As in the Solow model, with Cobb-Douglas technology

$$\hat{k}^* = \left(\frac{s}{n+g+\delta} \right)^{1/(1-\alpha)}$$

and

$$\hat{y}^* = \left(\frac{s}{n+g+\delta} \right)^{\alpha/(1-\alpha)} .$$

In this model capital per worker $k = e^{\phi\mu}A\hat{k}$ and output per worker $y = e^{\phi\mu}A\hat{y}$ so

$$k^* = e^{\phi\mu}A\left(\frac{s}{n+g+\delta}\right)^{1/(1-\alpha)}$$

and

$$y^* = e^{\phi\mu}A\left(\frac{s}{n+g+\delta}\right)^{\alpha/(1-\alpha)}.$$

Differences in output per capita across countries can be attributed in this model to differences in μ, the amount of education and training in the labour force.

3 Knowledge and Technical Progress

Models of endogenous growth make one of two assumptions about the generation of technical progress. In one branch of the literature, technical progress arises from research and development in a non-productive ideas sector. In the other branch, technical progress arises spontaneously through learning by doing.

3.1 Learning by doing

The idea of learning by doing stems from a seminal paper by Kenneth Arrow (Arrow (1962)). Arrow quotes the example of aircraft production where engineers had observed that the number of labour-hours needed to produce an airframe (the aircraft body without engines) is a decreasing function of the total number of airframes of the same type previously produced. (The precise relationship was found to be $L \propto N^{1/3}$ where L is labour-hours and N the Nth airframe). The idea is that workers become more efficient through gaining experience doing a job. This efficiency gain is an externality to the firm since it could be transferred if a worker moved to another firm. This idea of a *learning curve* was not new (Hirsch (1956) for example showed the existence of learning curves in different production processes) but Arrow was the first to build a model of growth using cumulative gross investment as an index for experience.

3.2 Knowledge spillovers

Ideas have an important attribute that distinguish them from other factors of production which is that they are *non-rivalous*. One firm using an idea does not prevent other firms from using the same idea. Of course patenting exists as a way to protect an idea and stop it from spreading freely but patents can often be circumvented and can only last for a limited time. Because it is difficult to prevent a good idea from spreading it can be thought of as having (positive) externalities. This means that even though each firm's production function exhibits constant returns to scale, there can be increasing returns to scale at the economy level due to the spillover of ideas.

4 Productive Externalities

4.1 The Romer 1986 Model

Romer (1986) follows Arrow (1962) and constructs a model in which learning by doing is related to the capital stock. This creates an externality leading to increasing returns to scale at the economy level. The production function is

$$Y = K^{\alpha}(AL)^{1-\alpha} \tag{4.1}$$

but technical progress A is assumed to be proportional to the capital stock K

$$A = K^{\phi} \tag{4.2}$$

where $\phi > 0$. The firm treats technical progress as given so that, at the firm level, there are constant returns to scale and the shares of income paid to capital and labour are α and $1-\alpha$ respectively. However, at the economy level, taking into account that technical progress is proportional to K, the production function is

$$Y = K^{\alpha}(K^{\phi}L)^{1-\alpha} = K^{\alpha+\phi(1-\alpha)}L^{1-\alpha} \tag{4.3}$$

which exhibits increasing returns to scale, with the exponents summing to $1 + \phi(1-\alpha) > 1$. Note that if $\phi < 1$, the exponent on K, $\alpha + \phi(1-\alpha) < 1$ so there are still diminishing marginal returns to capital but if $\phi = 1$ then there are constant marginal returns to capital and if $\phi > 1$ then marginal returns to capital are increasing.

Making the standard assumptions that

$$\dot{K} = I - \delta K \tag{4.4}$$

and

$$I = sY \tag{4.5}$$

we can derive the steady state growth path in this model.

4.2 The *AK* Model

A very simple but influential model can be obtained by assuming that technical progress is proportional, not to capital K itself but to capital per worker K/L. Replacing (4.2) with

$$A = a\frac{K}{L}, \tag{4.6}$$

where $a > 0$ is a constant, in (4.1) gives

$$Y = K^{\alpha}(a\frac{K}{L}L)^{1-\alpha} = a^{1-\alpha}K = \bar{A}K \tag{4.7}$$

where $\bar{A} = a^{1-\alpha} > 0$. This model is known as the *AK model.* This production function is *linear* and has constant returns to scale and non-diminishing returns to the factor of production K since

$$\frac{\partial Y}{\partial K} = \bar{A} > 0 \qquad \text{and} \qquad \frac{\partial^2 Y}{\partial K^2} = 0. \tag{4.8}$$

Nevertheless, at the firm level where technical progress A is taken as given, diminishing returns to K and L in (4.1) still hold.

The AK production function was probably first used by John von Neumann in 1937 (von Neumann (1945)) but came to prominence in Rebelo (1991). Dividing (4.7) by L gives

$$y = f(k) = \bar{A}k \tag{4.9}$$

and substituting into the standard Solow differential equation

$$\dot{k} = sf(k) - (n+\delta)k. \tag{4.10}$$

gives

$$\dot{k} = s\bar{A}k - (n+\delta)k = (s\bar{A} - n - \delta)k \tag{4.11}$$

or

$$\frac{\dot{k}}{k} = s\bar{A} - n - \delta. \tag{4.12}$$

Capital per worker is only constant in this model if $s\bar{A} = n + \delta$ which will only happen by chance. If $s\bar{A} > n + \delta$ then capital and output per worker will grow indefinitely. In particular, per capita growth will still occur even when the population is not growing so that $n = 0$.

5 Models of R&D

An alternative approach to endogenous growth assumes that technical progress, rather than being generated spontaneously by learning by doing, is the result of purposeful research and development. An example of this approach is the model of Paul Romer (1990). It is assumed that the economy comprises two sectors: the productive sector which uses physical capital and labour to produce output and an *R&D* sector that uses labour to produce ideas that are then used in the productive sector. The productive sector has a standard Cobb-Douglas production function

$$Y = K^{\alpha}(A\lambda L)^{1-\alpha} \tag{5.1}$$

where $0 < \lambda < 1$ is the proportion of the labour force employed in the productive sector and A is now the stock of ideas from the *R&D* sector.

The *R&D* sector employs $(1-\lambda)L$ workers to produce new ideas. It is assumed that the rate of change of ideas $\dot{A}$ is proportional to the number of workers employed in the sector so that

$$\dot{A} = \bar{\gamma}((1-\lambda)L)^{\beta} \tag{5.2}$$

where $\bar{\gamma} > 0$ is the rate of flow of ideas per researcher and $0 < \beta \leq 1$. Note that it is assumed in this simple model that capital is not required in the production of ideas. If $\beta = 1$ then the production function (5.2) exhibits constant returns to scale while if $\beta < 1$ there are decreasing returns to scale. An argument for why decreasing returns might be expected in the *R&D* sector is a negative externality at the aggregate level due to 'stepping on toes' from researchers in different companies working on the same new idea. However, Romer (1990) assumes $\beta = 1$.

We could assume that $\bar{\gamma}$ is constant but it is more plausible to suppose that it is related to the stock of ideas A. We will assume

$$\bar{\gamma} = \gamma A^{\psi} \tag{5.3}$$

where $\gamma > 0$. When $\psi = 0$ then the rate of flow of ideas per researcher is constant. When $\psi > 0$ the rate of ideas increases with the stock. This corresponds to the familiar 'standing on the shoulders of giants' quotation of Isaac Newton acknowledging that research builds on previous work. Conversely, if $\psi < 0$ then the rate of new ideas declines with the stock. This might be the case when A is very large so that the 'pond' of ideas is 'fished out'.

Substituting (5.3) into (5.2) and dividing by A gives the growth rate of A:

$$\frac{\dot{A}}{A} = g_A = \gamma A^{\psi-1}((1-\lambda)L)^{\beta}. \tag{5.4}$$

This growth rate depends on A. The growth rate of the growth rate of A is defined by

$$\frac{\dot{g_A}}{g_A} = (\psi - 1)\frac{\dot{A}}{A} + \beta\frac{\dot{L}}{L} = \beta n - (1-\psi)\frac{\dot{A}}{A} \tag{5.5}$$

and for a constant growth rate of ideas we require this to be zero which implies

$$\frac{\dot{A}}{A} = \frac{\beta n}{1-\psi} = g_A^*. \tag{5.6}$$

Note that this is independent of λ.

As long as $\psi < 1$, it follows that $1-\psi > 0$ and so this steady state growth rate will be positive. However, if $\psi > 1$ the steady state growth rate g_A^* is negative. Note that if the population stops growing so that $n = 0$, then the growth in ideas eventually stops also. The reason is that, if the number of researchers $(1-\lambda)L$ is constant, the rate of change of new ideas is constant (assume $\psi = 0$ for simplicity) which means that the *growth rate* of ideas is falling. Only if the population growth is positive is a positive growth rate of ideas sustainable.

Having determined the steady state growth rate of ideas A in this model, we can calculate the steady state growth path for capital, using the familiar differential equation for the Solow model with technical progress:

$$\dot{\hat{k}} = sf(\hat{k}) - (g_A + n + \delta)\hat{k}. \tag{5.7}$$

Dividing (5.1) by AL, the intensive production function for the productive sector is

$$\hat{y} = f(\hat{k}) = \hat{k}^{\alpha}\lambda^{1-\alpha} \tag{5.8}$$

and substituting (5.8) and (5.6) into (5.7) and setting to zero for steady state gives

$$s\hat{k}^{*\alpha}\lambda^{1-\alpha} = (\frac{\beta n}{1-\psi} + n + \delta)\hat{k}^* \tag{5.9}$$

or

$$\hat{k}^* = \lambda\left(\frac{s}{g_A + n + \delta}\right)^{1/(1-\alpha)} \tag{5.10}$$

and, from (5.8),

$$\hat{y}^* = \lambda\left(\frac{s}{g_A + n + \delta}\right)^{\alpha/(1-\alpha)}. \tag{5.11}$$

In steady state, both capital per worker $k = K/L$ and output per worker $y = Y/L$ grow at the steady state rate of growth of ideas g_A^*.

What is the optimal size of the *R&D* sector? Output per worker is $y = A\hat{y}$ and on the steady state path, from (5.4),

$$A = \left(\frac{\gamma((1-\lambda)L)^{\beta}}{g_A}\right)^{1/(1-\psi)} \tag{5.12}$$

$$= (1-\lambda)^{\beta/(1-\psi)} \left(\frac{\gamma L^{\beta}}{g_A}\right)^{1/(1-\psi)} \tag{5.13}$$

so that, from (5.11) and (5.13),

$$y = \lambda(1-\lambda)^{\beta/(1-\psi)} \left[\left(\frac{\gamma L^{\beta}}{g_A}\right)^{1/(1-\psi)} \left(\frac{s}{g_A + n + \delta}\right)^{\alpha/(1-\alpha)}\right]. \tag{5.14}$$

We are looking for the value of λ that maximises this expression. Differentiating with respect to λ and noting that the term in square brackets in (5.14) is independent of λ and so can be treated as constant gives

$$\frac{\partial y}{\partial \lambda} = \left((1-\lambda)^{\beta/(1-\psi)} - \frac{\lambda\beta}{1-\psi}(1-\lambda)^{\beta/(1-\psi)-1}\right)\left[\quad\cdot\quad\right] \tag{5.15}$$

$$= (1-\lambda)^{\beta/(1-\psi)} \left(1 - \frac{\lambda\beta}{(1-\psi)(1-\lambda)}\right)\left[\quad\cdot\quad\right]. \tag{5.16}$$

For this expression to be equal to zero (for a maximum) requires

$$1 - \frac{\lambda\beta}{(1-\psi)(1-\lambda)} = 0 \tag{5.17}$$

and solving for λ gives

$$\lambda = \frac{1-\psi}{1-\psi+\beta} \tag{5.18}$$

and

$$1-\lambda = \frac{\beta}{1-\psi+\beta}. \tag{5.19}$$

These expressions give the optimal proportions of the labour force in the production and ideas sectors that maximise output per head. Note that for $\beta = 1$ and $\psi = 0$ (the case where there are constant returns to scale in the ideas sector and the rate of flow of ideas per researcher is constant)

$$\lambda = 1 - \lambda = 0.5 \tag{5.20}$$

so that half the labour force is working in the *R&D* sector. This is a much larger proportion than we observe in the real world.

6 Conclusions

Endogenous growth models and human capital models share some common features and they both help account for differences in per capita growth rates in different countries. Next week we look at optimal growth models which attempt to provide a microeconomic foundation for why households in economies might rationally choose to save.

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