Macroeconomics
Lecture 5: Optimal Growth Models

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1 Introduction

The Solow-Swan model treats savings as a constant, exogenously given, proportion of income. Though we saw in Lecture 1 that, in that model there is a savings rate s that maximises consumption per worker defined by the *golden rule of accumulation*

$$f'(k^*(s)) = n + \delta, \tag{1.1}$$

it is not clear that this is an *optimal* savings rate since no utility function for society has been specified. It is also not clear that the optimal savings rate should always be constant.

The aim of this lecture is to examine reasons for savings and describe two models in which savings decisions are made by rational households and firms to optimise an explicit utility function. In this sense the models are *micro-founded* models of growth.

2 The Reasons for Saving

Saving is a way of postponing consumption from the present until some future period. To understand why consumers might choose to save requires exploring their intertemporal preferences. In general, preferences over consumption bundles at different points in time can be represented by a utility function

$$U(c_1, c_2, c_3, \cdots) \tag{2.1}$$

where c_1 is per capita consumption by a household in period t_1 and c_2 is per capita consumption in period t_2 etc. (Writing the utility function in terms of per capita consumption of the household avoids having to introduce the size of the household later when we aggregate over households to the whole population). Consider the two-period case

$$U(c_1, c_2). \tag{2.2}$$

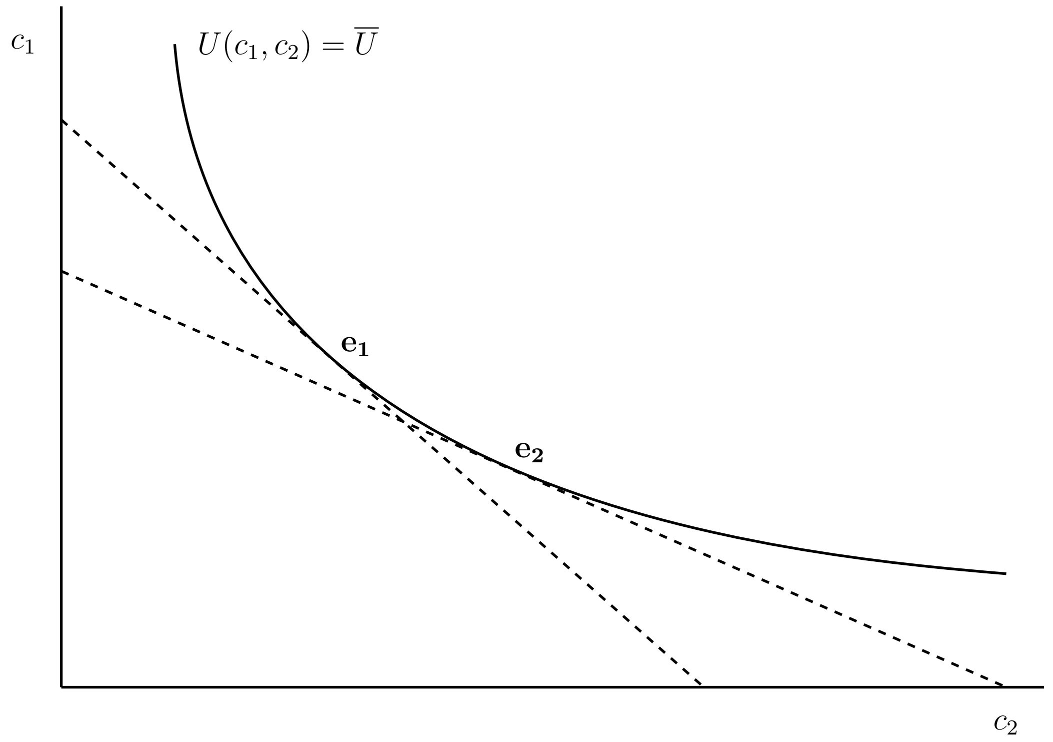

Figure 1: Intertemporal indifference curve: $U(c_1, c_2) = \overline{U}$

The intertemporal indifference curves of the utility function are given by

$$U(c_1, c_2) = \overline{U} \tag{2.3}$$

and are represented in Figure 1. The slope of the indifference curves is

$$Q = \frac{\partial U/\partial c_1}{\partial U/\partial c_2} = \frac{U_{c_1}}{U_{c_2}}. \tag{2.4}$$

The intertemporal elasticity of substitution is defined by

$$-\frac{d(c_1/c_2)}{d(U_{c_1}/U_{c_2})}\frac{U_{c_1}/U_{c_2}}{c_1/c_2} = -\left[\frac{\partial Q}{\partial(c_1/c_2)}.\frac{c_1/c_2}{Q}\right]^{-1}. \tag{2.5}$$

which is a measure of the curvature of the intertemporal indifference curves. (Note the close analogy with the elasticity of substitution in the production function from Lecture 1). Taking the limit of (2.5) as $c_1 \to c_2$ gives the *instantaneous elasticity of substitution* defined by

$$-\frac{\partial U/\partial c}{\partial^2 U/\partial c^2}\frac{1}{c}. \tag{2.6}$$

For the case of a Cobb-Douglas utility function

$$U(c_1, c_2) = c_1^{\alpha} c_2^{\beta} \tag{2.7}$$

we have

$$Q = \frac{U_{c_1}}{U_{c_2}} = \frac{\alpha U / c_1}{\beta U / c_2} = \frac{\alpha / \beta}{c_1 / c_2} \tag{2.8}$$

and

$$\frac{\partial Q}{\partial (c_1/c_2)} = -\frac{Q}{c_1/c_2}. \tag{2.9}$$

For this case the intertemporal elasticity of substitution is

$$-\left[\frac{\partial Q}{\partial (c_1/c_2)} . \frac{c_1/c_2}{Q}\right]^{-1} = -\left[-\frac{Q}{c_1/c_2} . \frac{c_1/c_2}{Q}\right]^{-1} = 1 \tag{2.10}$$

which is constant.

Note that in the Cobb-Douglas utility function (2.7), the parameters α and β represent the weightings given to consumption in periods t_1 and t_2 respectively so that the ratio α/β reflects the consumer's rate of time preference. Taking logarithms of (2.7) gives

$$\log U = \hat{U} = \alpha \log c_1 + \beta log c_2 \tag{2.11}$$

which shows that the utility function displays additive time separability (taking logarithms is a monotonic transformation of the utility function and so does not change its fundamental properties or household behaviour).

A time-additive utility function can be generalised to many periods $t = 0, \cdots, T$, where T is the time horizon, to give

$$U = \sum_{t=0}^{T} \alpha_t u(c_t) \tag{2.12}$$

in discrete time or

$$U = \int_0^T \alpha_t u(c_t) dt \tag{2.13}$$

in continuous time, where $u(c_t)$ is called the *instantaneous utility function* and represents the utility from consumption at period t and where α_t are the time varying weights on $u(c_t)$. Properties that we expect of the instantaneous utility function $u(\cdot)$ are that $u'(\cdot) > 0$ and $u''(\cdot) < 0$. This ensures that the function is increasing in c and concave. We also assume that $u(\cdot)$ satisfies the Inada conditions that $u'(c) \to \infty$ as $c \to 0$ and $u'(c) \to 0$ as $c \to \infty$.

For the Cobb-Douglas case, the instantaneous utility function is

$$u(c_t) = \log c_t. \tag{2.14}$$

A more general case is the *constant intertemporal elasticity of substitution* (*CIES*) utility function which is defined by

$$u(c_t) = \frac{c_t^{1-\theta}}{1-\theta} \tag{2.15}$$

where $\theta > 0$ and $\sigma = 1/\theta$ is the elasticity of substitution. The smaller is θ, the more willing is the household to vary its consumption over time. When θ is close to zero, the utility function is almost linear in c_t so the household will accept big variations in consumption over time. Using l'Hôpital's rule it can be shown that the Cobb-Douglas is a special case of the *CIES* utility function where $\theta = 1$. It is sometimes known as the Brock-Mirman utility function after its use in Brock and Mirman (1972).

A standard assumption to make in (2.12) or (2.13) is that the weights α_t decline geometrically with time so that

$$\alpha_t = e^{-\rho t} \tag{2.16}$$

where $\rho \geq 0$ is the *rate of time discount*. The case $\rho = 0$ is where there is no time discounting so that weights $\alpha_t = 1$ for all t. Ramsey himself assumed $\rho = 0$ because he saw the problem as one of a social planner rather than of individual households and viewed discounting the utility of future generations as 'ethically indefensible'. However, the more usual assumption is that $\rho > 0$ which implies that future consumption is less valued than present consumption in the utility function. In a world of uncertainty, postponing consumption to the future is risky and so a value of ρ greater than zero can be justified as an aversion to risk. In a world of perfect certainty, this assumption is less easy to rationalise but must simply be accepted as a 'selfish' preference.

An alternative assumption to (2.16) sometimes used in discrete time is

$$\alpha_t = (1+\rho)^{-t} \tag{2.17}$$

which is approximately equal to (2.16) for small ρ.

Combining (2.17) with (2.12) or (2.16) with (2.13) gives

$$U = \sum_{t=0}^{T} (1+\rho)^{-t} u(c_t) \tag{2.18}$$

for discrete time or

$$U = \int_0^T e^{-\rho t} u(c_t) dt \tag{2.19}$$

for the continuous time case.

3 The Ramsey-Cass-Koopmans Model

This section follows Blanchard and Fisher (1989) which uses a simpler version of the Ramsey model than Romer (2012). The principal differences are (i) the utility function is formulated per capita to avoid household size appearing in the problem (ii) we assume no technical progress, $g = 0$, so variables c, k and y are *per capita* rather than *per effective worker* as in Romer.

The optimal savings model of Ramsey (1928) was further developed by Cass (1965) and Koopmans (1965) and is now generally known as the *Ramsey-Cass-Koopmans* model. The model assumes an infinite time horizon $T = \infty$ so that each household maximises

$$U = \int_0^\infty e^{-\rho t} u(c(t))dt. \tag{3.1}$$

The capital stock accumulates according to the familar equation

$$\dot{K} = I + \delta K \tag{3.2}$$

except that for simplicity we assume no capital depreciation so $\delta = 0$ and $\dot{K} = I$. Substituting this expression into the income identity

$$Y = C + I = C + \dot{K} \tag{3.3}$$

and dividing by L gives

$$y = f(k) = c + \frac{\dot{K}}{L} = c + \dot{k} + nk \tag{3.4}$$

where the first equality is the intensive production function and the last equality uses the fact that

$$\dot{k} = \frac{\dot{K}}{L} - k\frac{\dot{L}}{L} \tag{3.5}$$

and the assumption that L grows at a constant rate n. Capital per worker in the first period $k(0)$ is some known value $\overline{k} > 0$.

Households are assumed to choose the path of per capita consumption $c(t)$ (which since $c = y - k$ means choosing $k(t)$) to maximise utility (3.1) subject to the budget constraint (3.4), the initial condition $k(0) = \overline{k}$ and the non-negativity constraints that $c(t), k(t) \geq 0$ for all t. Substituting (3.4) into (3.1), the problem to be solved can formally be stated as

$$\max_k U = \int_0^\infty e^{-\rho t} u(f(k) - \dot{k} - nk)dt. \tag{3.6}$$

It can be seen that utility is a function of both k and its time derivative $\dot{k}$. This means that the problem cannot be solved using conventional calculus but requires the use of either the *calculus of variations*, *dynamic programming* or Pontryagin's *maximal principle of optimal control.*

We will use the *maximal principle.* This solves problems of the form

$$\max_x V = \int_0^T F(y, x, t)dt$$

subject to

$$\dot{y} = f(y, x, t)$$

and

$$y(0) = \overline{y}$$

with $\overline{y}$ known. The variable y is known as the *state variable* and x the *control variable.* Solution involves forming the *Hamiltonian*

$$H(y, x, t, \lambda) = F(y, x, t) + \lambda(t)f(y, x, t).$$

The Hamiltonian is rather like the Lagrangian in a standard constrained optimisation with $\lambda(t)$ corresponding to a Lagrangian multiplier except that here there is a continuum of them that can vary over time. The λs are known as dynamic Lagrangian multiplers or *costate* variables. The *maximal principle* states that the conditions for the solution of the optimisation problem are

$$\frac{\partial H}{\partial x} = 0, \text{ for all } t,\ t = 1, \cdots, T$$

$$\frac{\partial H}{\partial \lambda} = \dot{y}$$

$$\frac{\partial H}{\partial y} = -\dot{\lambda}$$

and

$$\lambda(T)y(T) = 0.$$

The last condition is known as the *transversality condition.* For infinite horizon problems where $T = \infty$ this becomes

$$\lim_{t\to\infty} \lambda(t)y(t) = 0.$$

The Ramsey problem can be written as

$$\max_c U = \int_0^\infty e^{-\rho t}u(c(t))dt \tag{3.7}$$

subject to

$$\dot{k} = f(k) - c - nk \tag{3.8}$$

and

$$k(0) = \overline{k} \tag{3.9}$$

so that if we take k as the state variable and c as the control variable then the Hamiltonian can be written as

$$H(k, c, t, \lambda) = e^{-\rho t}u(c(t)) + \lambda(t)(f(k) - c - nk). \tag{3.10}$$

The costate variables $\lambda(t)$ can be interpreted as the marginal value at time 0 of an additional unit of capital at time t.

The four conditions for a maximum are

$$\frac{\partial H}{\partial c} = e^{-\rho t}u'(c) - \lambda = 0 \tag{3.11}$$

$$\frac{\partial H}{\partial \lambda} = f(k) - c - nk = \dot{k} \tag{3.12}$$

$$\frac{\partial H}{\partial k} = \lambda(f'(k) - n) = -\dot{\lambda} \tag{3.13}$$

and

$$\lim_{t\to\infty} \lambda(t)k(t) = 0. \tag{3.14}$$

Differentiating (3.11) with respect to time gives

$$e^{-\rho t}\frac{du'(c)}{dt} - \rho e^{-\rho t}u'(c) - \dot{\lambda} = 0 \tag{3.15}$$

and, combining with (3.13) and (3.11), the costate variables λ can be eliminated to give

$$e^{-\rho t}\frac{du'(c)}{dt} - \rho e^{-\rho t}u'(c) = -e^{-\rho t}u'(c)(f'(k) - n) \tag{3.16}$$

or, rearranging,

$$\frac{du'(c)}{dt} = (\rho + n - f'(k))u'(c). \tag{3.17}$$

This is known as an *Euler equation* since it corresponds to an equation developed by the Swiss mathematician Leonhard Euler (1707-1783) for the solution of problems in the calculus of variations. Noting that

$$\frac{du'(c)}{dt} = \frac{du'(c)}{dc}\frac{dc}{dt} = u''\dot{c} = u''c\frac{\dot{c}}{c}, \tag{3.18}$$

the Euler equation (3.17) can be rewritten as

$$\left(\frac{u''c}{u'}\right)\frac{\dot{c}}{c} = \rho - f'(k) + n = \rho - r + n \tag{3.19}$$

where the last equality comes from assuming that capital is paid its marginal product so that $f'(k) = r$ where r is the rate of return to capital (or savings).

Equation (3.19) is known as the *Ramsey rule for savings* or the *Keynes-Ramsey rule* (because Keynes first helped interpret this result). The term in brackets on the left-hand side of (3.19) is minus the reciprocal of the instantaneous elasticity of substitution derived in (2.5) and it must be negative. For a utility function in the class of *constant elasticity of intertemporal substitution* (*CIES*) utility functions (2.15), it will also be constant. In general, the optimal growth rate in per capita consumption will be positive as long as the rate of return on capital (or savings) r exceeds the rate of time discount ρ. This is an intuitive result. The higher the rate of return compared with the rate of time discount, the more willing are consumers to forego consumption today to enjoy consumption in the future.

The Ramsey model allows a balanced growth path where $\dot{c}/c$ is constant *only* for utility functions in the *CIES* class for which the term in brackets in (3.19) is constant. For the *CIES* utility function

$$u(c) = \frac{c^{1-\theta}}{1-\theta} \tag{3.20}$$

we have $u' = c^{-\theta}$ and $u'' = -\theta c^{-\theta-1}$ so that

$$\frac{u''c}{u'} = \frac{-\theta c^{-\theta-1}c}{c^{-\theta}} = -\theta \tag{3.21}$$

and the Ramsey rule becomes

$$\frac{\dot{c}}{c} = \frac{f'(k) - \rho - n}{\theta}. \tag{3.22}$$

In steady state, $\dot{c} = \dot{k} = 0$, $k = k^*$ and so from (3.4)

$$c^* = f(k^*) - nk^* \tag{3.23}$$

and the Ramsey rule (3.22) implies

$$f'(k^*) = \rho + n. \tag{3.24}$$

Compare this rule with the *golden rule of capital accumulation* from the Solow model of Lecture 1

$$f'(k^*) = n + \delta \tag{3.25}$$

or

$$f'(k^*) = n \tag{3.26}$$

without capital depreciation. The difference between the Ramsey optimal rule and the Solow golden rule is the rate of time discount in the utility function ρ which appears in the Ramsey rule. Actually, Ramsey himself assumed $\rho = 0$ so that his optimal growth path in steady state is the same as the golden rule.

However, the Ramsey rule is much more general. It defines the optimal rate of growth both in and out of steady state and for general utility functions for which the elasticity of intertemporal consumption may not be constant (so that balanced growth cannot exist). Consider the utility function

$$u(c) = -\frac{1}{\alpha}e^{-\alpha c} \tag{3.27}$$

where $\alpha > 0$. For this utility function

$$\frac{u''c}{u'} = \frac{-\alpha e^{-\alpha c}c}{e^{-\alpha c}} = -\alpha c \tag{3.28}$$

so the instantaneous elasticity of substitution is $1/(\alpha c)$ which is a decreasing function of consumption. With this utility function, the Ramsey rule (3.19) becomes

$$\dot{c} = \frac{f'(k) - \rho - n}{\alpha} \tag{3.29}$$

so that the change in consumption is proportional to the excess of the marginal product of capital (net of population growth) over the discount rate. The utility function (3.27) is known as the *constant absolute risk aversion* utility function.

What is the meaning of the transversality condition (3.14)? Combining (3.14) with (3.11) gives

$$\lim_{t\to\infty} e^{-\rho t}u'(c)k(t) = 0. \tag{3.30}$$

This condition is easier to interpret for the finite horizon case where it becomes

$$e^{-\rho T}u'(c)k(T) = 0. \tag{3.31}$$

Now $e^{-\rho T}u'(c)$ is the present (discounted) value of marginal utility in the final period T. If this were positive then it would not be optimal to end up with a positive capital stock $k(T)$ since this could have been consumed instead. The condition (3.30) or (3.31) also rules out so-called *Ponzi schemes*. Charles Ponzi was a swindler from Boston who became rich through a postal scam in which he borrowed money from investors and repaid them by borrowing ever more money. He was finally imprisoned and died a pauper. In this model, a Ponzi scheme would involve some households borrowing from other households in order to finance consumption above their lifetime income. With a finite number of households but an infinite time horizon, Ponzi schemes are not feasible though they can arise in other models unless ruled out by a transversality condition.

4 The Diamond Overlapping Generations Model

In the Ramsey model we have infinitely lived households who are concerned with maximising utility over an infinite horizon. An alternate approach is to assume individuals who live for a finite time, working in the first part of their lives and retired in the last part and living off the savings they accumulated while working. In this model, the reason for savings is not to provide for future generations but to provide for retirement. In the economy at any point in time there are always overlapping generations of workers and the retired.

In the simplest case, Diamond (1965), we assume individuals live for just two periods. This simplification makes the model more tractable without materially affecting the results. Given the nature of the model, it is more logical to work in discrete time. We use lowercase letters c and y to denote individual consumption and income to distinguish from aggregate consumption and income for which we will use uppercase letters as usual. The lifetime utility function for an individual living in periods t and $t+1$ is

$$U_t = u(c_{1,t}) + \frac{1}{1+\rho} u(c_{2,t+1}) \tag{4.1}$$

where $c_{1,t}$ is consumption by the individual in the first (working) period and $c_{2,t+1}$ is consumption in the second (retired) period and the weights follow (2.17) where ρ is the rate of time discount.

The budget constraints of the individual in periods t and $t+1$ are defined by

$$y_t = c_{1,t} + s_t \tag{4.2}$$

and

$$c_{2,t+1} = (1 + r_{t+1}) s_t \tag{4.3}$$

where s_t is savings by the individual in period t and r_{t+1} is the rate of interest paid in period $t+1$ on savings invested for one period in t. These assumptions imply that the individual leaves no bequests (though this assumption can be relaxed). Substituting for s_t from (4.3) into (4.2), the lifetime budget constraint can be written as

$$y_t = c_{1,t} + \frac{1}{1+r_{t+1}} c_{2,t+1}. \tag{4.4}$$

The individual maximises lifetime utility (4.1) subject to the budget constraint (4.4). Specifically, the individual solves the Lagrangian problem

$$\ell = \max_{c_{1,t}, c_{2,t+1}} u(c_{1,t}) + \frac{1}{1+\rho} u(c_{2,t+1}) + \lambda (y_t - c_{1,t} - \frac{1}{1+r_{t+1}} c_{2,t+1}). \tag{4.5}$$

First order conditions for a maximum are

$$\frac{\partial \ell}{\partial c_{1,t}} = u'(c_{1,t}) - \lambda = 0 \tag{4.6}$$

$$\frac{\partial \ell}{\partial c_{2,t+1}} = \frac{1}{1+\rho} u'(c_{2,t+1}) - \lambda(\frac{1}{1+r_{t+1}}) = 0 \tag{4.7}$$

and

$$\frac{\partial \ell}{\partial \lambda} = y_t - c_{1,t} - \frac{1}{1+r_{t+1}} c_{2,t+1} = 0. \tag{4.8}$$

Substituting (4.7) into (4.6) gives

$$u'(c_{1,t}) - \frac{1+r_{t+1}}{1+\rho} u'(c_{2,t+1}) = 0 \tag{4.9}$$

or

$$\frac{u'(c_{1,t})}{u'(c_{2,t+1})} = \frac{1+r_{t+1}}{1+\rho} \tag{4.10}$$

which is the Euler equation for this model.

For the *CIES* utility function

$$u_t = \frac{c_t^{1-\theta}}{1-\theta} \tag{4.11}$$

$u' = c_t^{-\theta}$ so that (4.10) becomes

$$\frac{c_{1,t}^{-\theta}}{c_{2,t+1}^{-\theta}} = \frac{1+r_{t+1}}{1+\rho} \tag{4.12}$$

or

$$\frac{c_{2,t+1}}{c_{1,t}} = \left(\frac{1+r_{t+1}}{1+\rho}\right)^{1/\theta}. \tag{4.13}$$

From (4.2) and (4.3)

$$\frac{c_{2,t+1}}{c_{1,t}} = \frac{(1+r_{t+1})s_t}{y_t - s_t} \tag{4.14}$$

or, solving for s_t,

$$s_t = y_t \left(\frac{1+r_{t+1}}{c_{2,t+1}/c_{1,t}} + 1\right)^{-1}. \tag{4.15}$$

The term in brackets is positive so that, unambiguously,

$$\frac{\partial s_t}{\partial y_t} > 0 \tag{4.16}$$

and, savings increases with income. However, the sign of the derivative

$$\frac{\partial s_t}{\partial r_{t+1}} \tag{4.17}$$

is not unambiguous. Increasing the interest rate decreases the price of second-period consumption, leading individuals to shift consumption from the first to the second period. However, it also increases the feasible consumption set, allowing individuals to increase consumption in both periods. If the intertemporal elasticity of substitution is greater than one, then the substitution effect dominates and an increase in interest rates leads to an increase in saving.

For the *CIES* case

$$s_t = y_t \left(1 + (1 + r_{t+1})^{1-1/\theta}(1+\rho)^{1/\theta}\right)^{-1}. \tag{4.18}$$

Having examined the optimisation problem for an individual, we now need to consider the economy as a whole. The labour force born and working in period t, L_t, grows at the constant rate

$$L_t = (1+n)L_{t-1} \tag{4.19}$$

where n is a constant. Note that in this model the population N_t is not equal to the labour force but is equal to the sum of the labour force in this period (current workers) plus the labour force in the previous period (the retired):

$$N_t = L_t + L_{t-1} = \frac{2+n}{1+n} L_t \tag{4.20}$$

In the model savings are made by the L_t current workers while retired people use up their savings. This means that aggregate capital at the start of period $t+1$ K_{t+1} is equal to the savings of current workers in period t

$$K_{t+1} = S_t = L_t s_t. \tag{4.21}$$

Note that K_{t+1} is defined to be the capital stock at the beginning of period $t+1$ while savings S_t is a flow over the period t. Dividing equation (4.21) by L_t and using (4.19) gives

$$\frac{K_{t+1}}{L_t} = \frac{K_{t+1}}{L_{t+1}} \frac{L_{t+1}}{L_t} = (1+n)k_{t+1} = s_t \tag{4.22}$$

or

$$k_{t+1} = \frac{s_t}{(1+n)} \tag{4.23}$$

where $k_{t+1} = K_{t+1}/L_{t+1}$.

Aggregate output is produced by workers according to the production function

$$Y_t = F(K_t, L_t) \tag{4.24}$$

where we assume no technical progress. Assuming constant returns to scale, the production function (4.24) can be written in per worker form as

$$y_t = f(k_t) \tag{4.25}$$

where $y_t = Y_t/L_t$ and $k_t = K_t/L_t$. Assuming that capital is paid its marginal product

$$r_t = f'(k_t). \tag{4.26}$$

For the case of the *CIES* utility function, (4.18) gives an expression for the optimal rate of savings s_t in terms of y_t, r_{t+1} and the model parameters. Substituting (4.25) and (4.26) into (4.18) gives

$$s_t = f(k_t)\left(1 + (1 + f'(k_{t+1}))^{1-1/\theta}(1+\rho)^{1/\theta}\right)^{-1} \tag{4.27}$$

and, from (4.23)

$$k_{t+1} = (1+n)^{-1} f(k_t)\left(1 + (1 + f'(k_{t+1}))^{1-1/\theta}(1+\rho)^{1/\theta}\right)^{-1}. \tag{4.28}$$

which is the fundamental law of motion for capital in this overlapping generations model. For a steady state, we require $k_t = k_{t+1} = k^*$ in which case

$$k_t^* = (1+n)^{-1} f(k_t^*)\left(1 + (1 + f'(k_t^*))^{1-1/\theta}(1+\rho)^{1/\theta}\right)^{-1}. \tag{4.29}$$

5 Conclusions

The Ramsey-Cass-Koopmans and the Diamond overlapping generations model provide alternative frameworks in which agents choose optimally how much to save on the basis of explicit utility functions. In this sense they are both micro-founded models of growth unlike those we have looked at in previous lectures. Both models have rich behaviour but the cost of this richness is the added complexity of finding solutions to these models. Nevertheless, both models have been widely used in the literature.

This completes the survey of growth models in this course of lectures. Next week we switch to looking at short-run issues of cyclical behaviour in the context of a Keynesian model of the economy.

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