Structural Time Series Models

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1 Trend and Cycle Decomposition

$$\mathbf{y}_t = \boldsymbol{\mu}_t + \boldsymbol{\psi}_t$$

where $\mathbf{y}_t$ is an $n \times 1$ vector and $\boldsymbol{\mu}_t$ and $\boldsymbol{\psi}_t$ represent *trend* and *cycle* components respectively. This decomposition into components is *not unique*.

1.1 Beveridge-Nelson-Stock-Watson Decomposition

Beveridge and Nelson (1981) and Stock and Watson (1988) derive the following decomposition:

$$\begin{aligned}\Delta\mathbf{y}_t &= \mathbf{C}(L)\boldsymbol{\varepsilon}_t \\ &= \mathbf{C}(1)\boldsymbol{\varepsilon}_t + (1-L)\mathbf{C}^*(L)\boldsymbol{\varepsilon}_t\end{aligned}$$

Integrating up gives:

$$\mathbf{y}_t = \underbrace{\mathbf{C}(1)\sum_{i=0}^{\infty}\boldsymbol{\varepsilon}_{t-i}}_{\text{trend}} + \underbrace{\mathbf{C}^*(L)\boldsymbol{\varepsilon}_t}_{\text{cycle}}$$

NB:

(1) The 'cycle' here could be white noise or a first order AR process

$$\Delta y_t = \rho\Delta y_{t-1} + \nu_t$$

(2) Innovations in trend and cycle are *perfectly correlated*.

1.2 Structural Time Series Model

Structural Time Series Models, or Unobserved Components models, have been used by Harvey (1985), Watson (1986), and Koopman *et al.* (1995). We follow the general model of Koopman *et al.* which is implemented in the computer package *STAMP* (Version 5)

$$\mathbf{y}_t = \boldsymbol{\mu}_t + \boldsymbol{\psi}_t + \boldsymbol{\gamma}_t + \boldsymbol{\varepsilon}_t$$

where $\mathbf{y}_t$ is an $n \times 1$ vector and $\boldsymbol{\mu}_t$, $\boldsymbol{\psi}_t$, $\boldsymbol{\gamma}_t$ and $\boldsymbol{\varepsilon}_t$ represent *trend, cycle, seasonal, and irregular* components respectively where the irregular component $\boldsymbol{\varepsilon}_t$ with covariance matrix $\boldsymbol{\Sigma}_\varepsilon$ is not explained by the model. Separate unobserved components models are built of each of the components.

1.2.1 Trend component

The trend component is modelled by the local linear trend

$$\boldsymbol{\mu}_t = \boldsymbol{\mu}_{t-1} + \boldsymbol{\beta}_{t-1} + \boldsymbol{\eta}_t$$
$$\boldsymbol{\beta}_t = \boldsymbol{\beta}_{t-1} + \boldsymbol{\xi}_t$$

where $\boldsymbol{\mu}_t$ and $\boldsymbol{\beta}_t$ are $n \times 1$ vectors representing the *level* and *slope* of the trend respectively, and $\boldsymbol{\eta}_t$ and $\boldsymbol{\xi}_t$ are independent error processes with covariance matrices $\boldsymbol{\Sigma}_\eta$ and $\boldsymbol{\Sigma}_\xi$.

The stochastic slope parameter $\boldsymbol{\beta}_t$ allows the trend to *change smoothly* over time. With a stochastic slope, the trend process is second order integrated, $(I(2))$. However, in the special case where $\boldsymbol{\Sigma}_\xi$ is zero, the trend reduces to a random walk with drift given by $\boldsymbol{\beta}_t = \boldsymbol{\beta}_{t-1} = \overline{\boldsymbol{\beta}}$. If, in addition, $\boldsymbol{\Sigma}_\eta$ is zero, then the trend becomes deterministic.

1.2.2 Cycle component

The cycle component of the model is trigonometric in form and consists of one or more cycles defined by the pair of equations:

$$\begin{bmatrix} \boldsymbol{\psi}_t \\ \overline{\boldsymbol{\psi}}_t \end{bmatrix} = \rho \begin{bmatrix} \cos\lambda\mathbf{I}_n & \sin\lambda\mathbf{I}_n \\ -\sin\lambda\mathbf{I}_n & \cos\lambda\mathbf{I}_n \end{bmatrix} \begin{bmatrix} \boldsymbol{\psi}_{t-1} \\ \overline{\boldsymbol{\psi}}_{t-1} \end{bmatrix} + \begin{bmatrix} \boldsymbol{\omega}_t \\ \overline{\boldsymbol{\omega}}_t \end{bmatrix}$$

where $\boldsymbol{\psi}_t$ and $\overline{\boldsymbol{\psi}}_t$ are $n \times 1$ vectors and $\boldsymbol{\omega}_t$ and $\overline{\boldsymbol{\omega}}_t$ are vector error processes independent of $\boldsymbol{\varepsilon}_t$, $\boldsymbol{\eta}_t$ and $\boldsymbol{\xi}_t$ and with the same covariance matrix $\boldsymbol{\Sigma}_\omega = \Sigma_{\overline{\omega}}$, $\mathbf{I}_n$ is the identity matrix of dimension n and the vector process $\overline{\boldsymbol{\psi}}_t$ appears by construction.

The scalar parameters ρ and λ (which satisfy the restrictions $0 < \rho < 1$ and $0 < \lambda < \pi$) represent the cycle damping factor and frequency respectively.

We can write

$$\boldsymbol{\psi}_t - 2\rho\cos\lambda\boldsymbol{\psi}_{t-1} + \rho^2\boldsymbol{\psi}_{t-2} = \boldsymbol{\omega}_t - \rho\cos\lambda\boldsymbol{\omega}_{t-1} + \rho\sin\lambda\overline{\boldsymbol{\omega}}_{t-1}$$

which is a *restricted* $VARMA(2,1)$ model. Note that the parameters are the same for each variable. This corresponds to the assumption that the cycles are similar, and implies that the cycle for each variable has the same frequency λ and damping factor ρ, and identical autocorrelation function and spectrum.

The assumption that cycles are similar is a very strong one and imposes quite a serious restriction on the model. It does have the advantage of limiting the number of parameters that have to be estimated.

1.2.3 Seasonal component: (Dummy)

$$\boldsymbol{\gamma}_t = -\boldsymbol{\gamma}_{t-1} - \cdots - \boldsymbol{\gamma}_{t-s+1} + \boldsymbol{\zeta}_t$$

where the error process $\boldsymbol{\zeta}_t$ has covariance matrix $\boldsymbol{\Sigma}_\zeta$.

1.2.4 Seasonal component: (Trigonometric)

$$\boldsymbol{\gamma}_t = \sum_{j=1}^{[s/2]} \boldsymbol{\gamma}_{j,t}$$

and

$$\begin{bmatrix} \boldsymbol{\gamma}_{j,t} \\ \overline{\boldsymbol{\gamma}}_{j,t} \end{bmatrix} = \begin{bmatrix} \cos\lambda_j\mathbf{I}_n & \sin\lambda_j\mathbf{I}_n \\ -\sin\lambda_j\mathbf{I}_n & \cos\lambda_j\mathbf{I}_n \end{bmatrix} \begin{bmatrix} \boldsymbol{\gamma}_{j,t-1} \\ \overline{\boldsymbol{\gamma}}_{j,t-1} \end{bmatrix} + \begin{bmatrix} \boldsymbol{\zeta}_{j,t} \\ \overline{\boldsymbol{\zeta}}_{j,t} \end{bmatrix} \quad , \quad j = 1, \cdots, [s/2]$$

where $Var(\boldsymbol{\zeta}_{j,t}) = Var(\overline{\boldsymbol{\zeta}}_{j,t}) = \boldsymbol{\Sigma}_\zeta$, and $\lambda_j = 2\pi j/s$ is the frequency of the seasonal component.

1.2.5 The Hoderick-Prescott Model

This is a special case of the *univariate* Structural Time Series Model, with no cycle or seasonal component, defined by

$$y_t = \mu_t + \varepsilon_t$$

$$\mu_t = \mu_{t-1} + \beta_{t-1}$$

and

$$\beta_t = \beta_{t-1} + \xi_t$$

with the special restriction that $\sigma_\xi^2 = q\sigma_\varepsilon^2$, where the magic number $q = 1/1600$.

2 Common Factors

2.1 The Beveridge-Nelson-Stock-Watson Decomposition

$$\mathbf{y}_t = \mathbf{C}(1) \sum_{i=0}^{\infty} \boldsymbol{\varepsilon}_{t-i} + \mathbf{C}^*(L)\boldsymbol{\varepsilon}_t$$

Suppose there is cointegration among the n variables $\mathbf{y}_t$. This implies that

$$rank(\mathbf{C}(1)) < n$$

and that there exists a $n \times r$ matrix $\boldsymbol{\alpha}$ of *cointegrating vectors*, such that

$$\boldsymbol{\alpha}'\mathbf{C}(1) = 0 \ .$$

It also follows that $\mathbf{C}(1)$ can be decomposed into the product of two matrices

$$\mathbf{C}(1) = \boldsymbol{\delta}\ \boldsymbol{\beta}'$$

where $\boldsymbol{\delta}$ and $\boldsymbol{\beta}$ are both of dimension $n \times (n - r)$ and $\boldsymbol{\alpha}'\boldsymbol{\delta} = \mathbf{0}$. Then

$$\mathbf{C}(1) \sum_{i=0}^{\infty} \boldsymbol{\varepsilon}_{t-i} = \boldsymbol{\delta}\ \boldsymbol{\beta}' \sum_{i=0}^{\infty} \boldsymbol{\varepsilon}_{t-i} = \boldsymbol{\delta}\ \boldsymbol{\tau}_t$$

where $\boldsymbol{\tau}_t$ is a *vector of common trends*, and $\boldsymbol{\delta}$ is a *loading matrix* of coefficients on $\boldsymbol{\tau}_t$.

2.1.1 Common cycles

Similarly, if the variables exhibit common serial correlation features then, for some $(n \times s)$ matrix of *common features*, $\widetilde{\boldsymbol{\alpha}}$, it follows that

$$\widetilde{\boldsymbol{\alpha}}\ '\mathbf{C}^*(L) = \mathbf{0}$$

and that

$$\mathbf{C}^*(L) = \boldsymbol{\Phi}\ \boldsymbol{\lambda}(L)$$

where $\mathbf{\Phi}$ is an $(n \times n - s)$ loading matrix of coefficients with $\widetilde{\boldsymbol{\alpha}}\,'\mathbf{\Phi} = \mathbf{0}$ and $\boldsymbol{\lambda}(L)$ is an $(n - s \times n)$ matrix in the lag operator. Then

$$\mathbf{C}^*(L)\ \boldsymbol{\varepsilon}_t = \mathbf{\Phi}\ \boldsymbol{\lambda}(L)\ \boldsymbol{\varepsilon}_t = \mathbf{\Phi}\ \mathbf{c}_t$$

where $\mathbf{c}_t$ is a vector of *common cycles.*

The set of cofeature vectors $\widetilde{\boldsymbol{\alpha}}$ must be linearly independent of the cointegration vectors $\boldsymbol{\alpha}$ and it has to be the case that $r + s \leq n$.

Engle and Kozicki (1993) proposed a test for common cycles (features).This is analogous to the Johansen (1988) procedure for cointegrating vectors, except that the distribution of the statistics is standard because everything is stationary.

For the special case that $r + s = n$, Vahid and Engle (1993) show that the matrix

$$\mathbf{A} = \begin{bmatrix} \widetilde{\boldsymbol{\alpha}}' \\ \boldsymbol{\alpha}' \end{bmatrix}$$

is square and of full rank, with a conformably partitioned inverse defined by

$$\mathbf{A}^{-1} = \begin{bmatrix} \widetilde{\boldsymbol{\alpha}}^- & \boldsymbol{\alpha}^- \end{bmatrix} ,$$

and it follows that

$$\mathbf{y}_t = \mathbf{A}^{-1}\mathbf{A}\ \mathbf{y}_t = \widetilde{\boldsymbol{\alpha}}^-\widetilde{\boldsymbol{\alpha}}'\mathbf{y}_t + \boldsymbol{\alpha}^-\boldsymbol{\alpha}'\mathbf{y}_t = \mathbf{P}\ \boldsymbol{\delta}\ \boldsymbol{\tau}_t + (\mathbf{I} - \mathbf{P})\ \mathbf{\Phi}\ \mathbf{c}_t$$

where $\mathbf{P} \equiv \widetilde{\boldsymbol{\alpha}}^-\widetilde{\boldsymbol{\alpha}}'$ is an idempotent projection matrix and $\boldsymbol{\alpha}^-\boldsymbol{\alpha}' \equiv \mathbf{I} - \mathbf{P}$. Thus the condition $r + s = n$ allows a simple and unique decomposition into trend and cycle components that is independent of the normalisation of the cointegration and cofeature vectors.

2.2 Common trends and cycles in the Harvey model

Common trends and cycles can be incoroprated into the Structural Time Series Modelling framework, although there is no way to test for common trends and cycles within *STAMP*, so the number of cointegrating vectors, r, and cofeature vectors, s, have to be known *a priori.* Then the model, neglecting seasonal component, can be written as

$$\mathbf{y}_t = \mathbf{\Theta}_\mu \boldsymbol{\mu}_t^* + \overline{\boldsymbol{\mu}}^* + \mathbf{\Theta}_\psi \boldsymbol{\psi}_t^* + \boldsymbol{\varepsilon}_t$$

where $\mathbf{\Theta}_\mu$ and $\mathbf{\Theta}_\psi$ are *factor loading matrices* of dimensions $n \times (n - r)$ and $n \times (n - s)$ respectively, and $\overline{\boldsymbol{\mu}}^*$ is a fixed vector. In order to ensure identification, restrictions need to be imposed on these matrices. The restriction imposed in *STAMP* is that of lower triangularity so that

$$\boldsymbol{\Theta}_\mu = \begin{bmatrix} 1 & 0 & 0 & \cdots \\ \theta_{21} & 1 & 0 & \cdots \\ \theta_{31} & \theta_{32} & 1 & 0 \\ \vdots & \vdots & & \ddots \end{bmatrix}$$

and

$$\overline{\boldsymbol{\mu}}^* = \begin{bmatrix} \mathbf{0}_r \\ \overline{\boldsymbol{\mu}} \end{bmatrix}$$

where $\overline{\boldsymbol{\mu}}$ is an unrestricted $(n-r) \times 1$ vector.

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