Unit Roots

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1 Introduction

Definition 1 *Weak stationarity*
A variable y_t is weakly stationary if its mean and variance are constant over time.

Most economic variables do not satisfy the conditions of weak stationarity.

1.1 Two Simple Models

1.1.1 Deterministic Trend

$$y_t = \gamma + \beta t + \varepsilon_t \quad , \quad \varepsilon_t \sim iid(0, \sigma^2)$$

This model has a *non-constant* mean, and a *constant* variance. Stationarity is achieved by detrending.

1.1.2 Random walk with drift

$$y_t = c + y_{t-1} + \varepsilon_t \quad , \quad \varepsilon_t \sim iid(0, \sigma^2)$$

This model has both a *non-constant* mean, and a *non-constant* variance. Stationarity is achieved by first differencing. A series that can be made stationary by differencing is said to be *integrated*, or to possess a *unit root.*

Definition 2 *A time series y_t is integrated of order d, denoted $I(d)$, if $\Delta^d y_t$ is stationary. Then the series y_t has d unit roots.*

2 Testing for Unit Roots

Consider the model

$$y_t = \gamma + \beta t + u_t \tag{1}$$

and

$$u_t = (1-\alpha)u_{t-1} + \varepsilon_t \quad \varepsilon_t \sim iid(0, \sigma^2)\,. \tag{2}$$

A unit root test is a test of the null of

$$H_0 : \alpha = 0 \quad \text{against} \quad H_1 : \alpha > 0\,.$$

On H_0 :

$$\Delta y_t = \beta + \varepsilon_t$$

whereas, on H_1:

$$y_t = (1-\alpha)y_{t-1} + \alpha\beta t + (\gamma\alpha + (1-\alpha)\beta) + \varepsilon_t \tag{3}$$

or

$$\Delta y_t = -\alpha y_{t-1} + bt + c + \varepsilon_t\,. \tag{4}$$

2.1 The Dickey-Fuller (DF) Test

Fuller (1976) and Dickey and Fuller (1979, 1981) propose a test based on the t-ratio $t(\alpha)$ in the OLS regression (4). The distribution of this statistic is *non-standard* and depends on the presence of the *nuisance parameters*, β and γ. Two special cases need to be considered: (i) $\beta = 0 \Rightarrow b = 0$ and (ii) $\beta = \gamma = 0 \Rightarrow b = c = 0$. In the case (i), there is no drift term under the null H_0 and no trend term under the alternative H_1. In the case (ii) there is no drift term under the null H_0 and neither intercept nor trend term under the alternative H_1.

Critical values of the statistic for all three cases are given in Fuller (1976) Table 8.5.2 and in Banerjee *et al.* (1993).

2.2 The Durbin-Watson test

Sargan and Bhargava (1983) develop a test for a unit root based on the Durbin-Watson statistic in the equation (3). They show that, on the null hypothesis of a unit root, then

$$\operatorname*{plim}_{T\to\infty} DW = 0$$

Critical values for the test are given in Table 1 of Sargan and Bhargava (1983).

3 Dealing with autocorrelation

The tests in the previous section are based on the assumption that ε_t is 'white noise' *i.e. serially uncorrelated.* If ε_t is serially correlated then the serial correlation needs to be corrected before the unit root test is performed.

3.1 The Augmented Dickey-Fuller Test

This assumes that the serial correlation in ε_t can be represented by an $AR(p)$ process. Then it can be corrected by adding the p lagged terms $\Delta y_{t-1}, \cdots, \Delta y_{t-p}$ to the regression (4). The distribution of the test statistic is unaffected by the addition of these lagged differences.

3.2 The Phillips Z test

Phillips (1987) and Phillips and Perron (1988) propose a correction for serial correlation on the assumption that ε_t can be represented by an $MA(q)$ process. This involves making a *non-parametric* correction to the Dickey-Fuller test statistic based on the weighted estimated autocovariances. The Z-statistic is given by

$$Z(t_{\widehat{\alpha}}) = \frac{s}{s_*} t_{\widehat{\alpha}} - \frac{\frac{1}{2}(s_*^2 - s^2)}{s_* \sqrt{\frac{1}{T} |\mathbf{M}|}}$$

where $\mathbf{M} = \frac{1}{T}\mathbf{W}'\mathbf{W}$ is the moment matrix of the included regressors $\mathbf{W} = \{\mathbf{y}_{-1}, \boldsymbol{\iota}, \mathbf{t}\}$ in the model (4) and

$$s_*^2 = \frac{\widehat{\boldsymbol{\varepsilon}}'\widehat{\boldsymbol{\varepsilon}}}{T} + \sum_{i=1}^{q} w_i \frac{\widehat{\boldsymbol{\varepsilon}}'\widehat{\boldsymbol{\varepsilon}}_{-i}}{2T} .$$

is a Newey-West correction to the conventional error variance estimate s^2 with weights w_i. Different weighting schemes, (e.g. Bartlett, Parzen) can be used.

3.3 Said and Dickey test

Said and Dickey (1984) suppose that $\varepsilon_t \sim ARMA(p, q)$ with the moving average part invertible. In this case ε_t can be represented by an infinite order $AR(\infty)$ process. This justifies the use of the ADF test under more general conditions so long as the order of the estimated AR is long enough. Asymptotically, the order of the estimated AR process must $\rightarrow \infty$ as $T \rightarrow \infty$ but at a slower rate of order $< O(T^{\frac{1}{3}})$.

3.4 The Hall test

Hall (1989) proposes an alternative test for the assumption that $\varepsilon_t \sim MA(q)$. This is based on *Instrumental Variables* estimation using the variables y_{t-q-j} for $j > 0$ as valid instruments for $y_{t-1.}$ The test statistic follows the standard Dickey-Fuller distribution.

4 Testing unit roots versus structural breaks

The tests of the previous section consider the null hypothesis of a unit root against an alternative hypothesis of trend stationarity. Perron (1989) proposes testing the null against an alternative that allows for a structural break at a known point in time. This break could affect either the intercept, the trend slope or both under the alternative hypothesis. For the broken trend case, the maintained hypothesis replaces (1) with

$$y_t = \gamma + \beta t + \delta D_t + u_t \tag{5}$$

where D_t is a dummy variable taking the value 0 up to and including the break point t_b, and $t - t_b$ thereafter.

The null hypothesis of a unit root can then be tested by a t-test on the coefficient $\widehat{\alpha}$ in the regression:

$$\Delta y_t = -\alpha y_{t-1} + bt + c + dD_t + \varepsilon_t \,.$$

The critical values for this test are given in Perron (1989). Tests for more than one break can also been considered. Perron finds that discrimination between integrated and broken trend models is often difficult.

5 Higher Order Unit Roots

In the unit root tests considered above, the null hypothesis is that $H_0 : y_t \sim I(1)$ against the alternative that $H_1 : y_t \sim I(0)$. Suppose that y_t is actually $I(2)$? Then the previous test is invalid. A valid test would be a unit root test on the first difference ΔY_t with null

$$H_0 : \Delta y_t \sim I(1) \Rightarrow y_t \sim I(2)$$

and alternative

$$H_1 : \Delta y_t \sim I(0) \Rightarrow y_t \sim I(1) \,.$$

To conduct valid inference, it is important to test *sequentially downwards from the highest possible order of integration.* For example

1) Test *I(d)* against *I(d-1)*. If null is rejected then
2) Test *I(d-1)* against *I(d-2)*. If null is rejected then
3) Test *I(d-2)* against *I(d-3)*. etc.

6 Seasonal Unit Roots

Consider the quarterly seasonal process

$$\begin{aligned}\Delta_4 y_t &= (1-L^4)y_t = (1+L+L^2+L^3)(1-L)y_t \\ &= (1-L)(1+L)(1-iL)(1+iL)y_t\end{aligned} \tag{6}$$

which has *four* unit roots: one at frequency 0, and three at seasonal frequencies corresponding to cycles of 2 quarters, and 4 quarters (the pair of imaginary roots) respectively.

6.1 Dickey-Hasza-Fuller (*DHF*) test

Dickey, Hasza and Fuller (1984) derive a test of the hypothesis $\alpha_s = 0$ in the model

$$\Delta_s y_t = \alpha_s y_{t-s} + \varepsilon_t$$

against the alternative that $\alpha_s < 0$. The test statistic is simply the t-value on $\widehat{\alpha}_s$ and critical values for this test are presented in their paper (reprinted in Hylleberg (1992)) for the cases $s = 2$, 4, and 12. As with standard Dickey-Fuller tests, deterministic components (constant and trend) can be added to the specification but do affect the distribution of the statistic. Lagged values of $\Delta_s y_t$ can be added to 'whiten' the errors without affecting the distribution.

From (6) it can be seen that the *DHF(4)* test is a *joint test of four unit roots* against an alternative of no unit roots. In particular it tests for a unit root at zero frequency (i.e. the long run) at the same time as testing for seasonal unit roots.

6.2 Hylleberg-Engle-Granger-Yoo (*HEGY*) test

Hylleberg, Engle, Granger and Yoo (1990) develop a framework in which it is possible to *separately* test the four unit roots in (6) in the quarterly case ($s = 4$). This is based on constructing the model:

$$\begin{aligned}\Delta_4 y_t = \; &\pi_1(1+L+L^2+L^3)y_{t-1} \\ &-\pi_2(1-L+L^2-L^3)y_{t-1} \\ &-\pi_3(1-L^2)y_{t-2} - \pi_4(1-L^2)y_{t-1} + \varepsilon_t \,.\end{aligned}$$

The t-ratio on $\widehat{\pi}_1$ is a test of the null of a unit root at zero frequency and can be shown to follow a Dickey-Fuller distribution. The t-ratio on $\widehat{\pi}_2$ is a test of a unit root at the semi-annual frequency which also has a Dickey-Fuller distribution. The t-value on $\widehat{\pi}_3$ is a test for a unit root at the annual frequency, *conditional on the hypothesis that* $\pi_4 = 0$, and follows a *DHF(2)* distribution. Finally, a joint test of the hypothesis that $\pi_3 = 0$ and $\pi_4 = 0$ can be constructed from the F-statistic for a test of this restriction. The distribution of this last statistic is close to the standard $F_{2,T-k}$ distribution and critical values are tabulated in *HEGY*. As usual, adding lagged values of $\Delta_4 y_t$ to the regression does not change the distributions. However, if deterministic seasonal dummies are included in the regression, then this does affect the distribution of the tests of π_2, π_3, and π_4 leading to fatter tails.

The *HEGY* tests have been extended to the monthly case by Beaulieu and Miron (1993) and Franses (1991).

7 Further reading

For a very good concise treatment see Hamilton (1994) or, for more detail, read Banerjee *et al.* (1993). On seasonal unit roots, see Franses (1996a, 1996b) for a survey or look at the readings in Hylleberg (1992).

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